Groups and semigroups: connections and contrasts

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1 Introduction

Group theory and semigroup theory have developed in somewhat different directions in the past several decades. While Cayley's theorem enables us to view groups as groups of permutations of some set, the analogous result in semigroup theory represents semigroups as semigroups of functions from a set to itself. Of course both group theory and semigroup theory have developed significantly beyond these early viewpoints, and both subjects are by now integrally woven into the fabric of modern mathematics, with connections and applications across a broad spectrum of areas.

Nevertheless, the early viewpoints of groups as groups of permutations, and semigroups as semigroups of functions, do permeate the modern literature: for example, when groups act on a set or a space, they act by permutations (or isometries, or automorphisms, etc), whereas semigroup actions are by functions (or endomorphisms, or partial isometries, etc). Finite dimensional linear representations of groups are representations by invertible matrices, while finite dimensional linear representations of semigroups are representations by arbitrary (not necessarily invertible) matrices. The basic structure theories for groups and semigroups are quite different - one uses the ideal structure of a semigroup to give information about the semigroup for example - and the study of homomorphisms between semigroups is complicated by the fact that a congruence on a semigroup is not in general determined by one congruence class, as is the case for groups.

Thus it is not surprising that the two subjects have developed in somewhat different directions. However, there are several areas of modern semigroup theory that are closely connected to group theory, sometimes in rather surprising ways. For example, central problems in finite semigroup theory (which is closely connected to automata theory and formal language theory) turn out to be equivalent or at least very closely related to problems about profinite groups. Linear algebraic monoids have a rich structure that is closely related to the subgroup structure of the group of units, and this has interesting connections with the well developed theory of (von Neumann) regular semigroups. The theory of inverse semigroups (i.e. semigroups of partial one-one functions) is closely tied to aspects of geometric and combinatorial group theory.

In the present paper, I will discuss some of these connections between group theory and semigroup theory, and I will also discuss some rather surprising contrasts between the theories. While I will briefly mention some aspects of finite semigroup theory, regular semigroup theory, and the theory of linear algebraic monoids, I will focus primarily on the theory of inverse semigroups and its connections with geometric group theory.

For most of what I will discuss, there is no loss of generality in assuming that the semigroups under consideration have an identity - one can always just adjoin an identity to a semigroup if necessary - so most semigroups under consideration will be monoids, and on occasions the group of units (i.e. the group of invertible elements of the semigroup) will be of considerable interest.

2 Submonoids of groups

It is perhaps the case that group theorists encounter semigroups (or monoids) most naturally as submonoids of groups. For example, if P is a submonoid of a group G such that $P \cap P^{-1} = \{1\}$, then the relation $\leq_P$ on G defined by $g \leq_P h$ iff $g^{-1}h \in P$ is a left invariant partial order on G. This relation is also right invariant iff $g^{-1}Pg \subseteq P$ for all $g \in G$, and it is a total order iff $P \cup P^{-1} = G$. Note that $g \in P$ iff $1 \leq_P g$. Every left invariant partial order on G arises this way. One says that (G, P) is a partially ordered group with positive cone P. One may note that the partial order has the property that for all $g \in G$ there exists some $p \in P$ such that $g \leq_P p$ iff $G = PP^{-1}$, i.e. iff G is the group of (right) quotients of P. The study of ordered groups is well over a hundred years old, and I will not attempt to survey this theory here.

The question of embeddability of a semigroup (monoid) in a group is a classical question that has received a lot of attention in the literature. Clearly a semigroup must be cancellative if it is embeddable in a group. It is easy to see that commutative cancellative semigroups embed in abelian groups, in fact such a semigroup embeds in its group of quotients in much the same way as an integral domain embeds in a field. For non-commutative semigroups, the situation is far more complicated. One useful condition in addition to cancellativity that guarantees embeddability of a semigroup S in a group is the Ore condition. A semigroup S satisfies the *Ore condition* if any two principal right ideals intersect, i.e. $sS \cap tS \neq \emptyset$ for all $s, t \in S$. (In the language of many subsequent authors in the group theory literature, s and t have at least one *common multiple* for each $s, t \in S$: in the language of classical semigroup theory, one says that S is *left reversible*). The following well known result was essentially proved by Ore in 1931: a detailed proof may be found in Volume 1, Chapter 1 of the book by Clifford and Preston [30], which is a standard reference for basic classical results and notation in semigroup theory. There is an obvious dual result involving right reversible semigroups and groups of left quotients.

Theorem 1 *A cancellative semigroup satisfying the Ore condition can be embedded in a group. In fact a cancellative semigroup P can be embedded in a group $G = PP^{-1}$ of (right) quotients of P if and only if P satisfies the Ore condition.*

As far as I am aware, the first example of a cancellative semigroup that is not embeddable in a group was provided by Mal'cev in 1937 [101]. Necessary and sufficient conditions for the embeddability of a semigroup in a group were provided by Mal'cev in 1939 [102]. Mal'cev's conditions are countably infinite in number and no finite subset of them will suffice to ensure embeddability of a semigroup in a group. A similar set of conditions, with a somewhat more geometric interpretation, was provided by Lambek in 1951. Chapter 10 of Volume 2 of Clifford and Preston [30] provides an account of the work of Mal'cev and Lambek and a description of the relationship between the two sets of conditions.

The question of when a monoid with presentation $P = Mon\langle X : u_i = v_i\rangle$ embeds in a group has been studied by many authors, and has received attention in the contemporary literature in group theory. Clearly such a monoid embeds in a group if and only if it embeds in the group with presentation $G = Gp\langle X : u_i = v_i\rangle$. Here the u_i, v_i are *positive* words, i.e. $u_i, v_i \in X^*$, where X^* denotes the free monoid on X. We allow for the possibility that some of the words u_i or v_i may be empty, (i.e. the identity of X^*). Also, we use the notation $Mon\langle X : u_i = v_i\rangle$ for the monoid presented by the set X of generators and relations of the form $u_i = v_i$ to distinguish it from the group $Gp\langle X : u_i = v_i\rangle$ or the semigroup $Sgp\langle X : u_i = v_i\rangle$ with the same set of generators and relations. From an algorithmic point of view, the embeddability question is undecidable, as are many such questions about semigroup presentations or group presentations, since the property of being embeddable in a group is a Markov property (see Markov's paper [103]).

It is perhaps worth observing that being embeddable in a group is equivalent to *being* a group for special presentations where all defining relations are of the form $u_i = 1$. Recall that the *group of units* of a monoid P is the set

$$U(P) = \{a \in P : ab = ba = 1 \text{ for some } b \in P\}.$$

Proposition 1 *Let P be a monoid with presentation of the form $P = Mon\langle X : u_i = 1, i = 1 \dots n\rangle$, where each letter of X is involved in at least one of the relators u_i. Then P is embeddable in a group if and only if it is a group.*

Proof Suppose that P is embeddable in a group G, and consider a relation $u_i = 1$ in the set of defining relations of P. If $u_i = x_1 x_2 \dots x_n$ with each $x_j \in X$, then clearly x_1 is the inverse of $x_2 \dots x_n$ in G, so x_1 is in the group of units of P and $x_2 \dots x_n x_1 = 1$ in G also. It follows that x_2 is in the group of units of P, and similarly each x_j must be in the group of units of P. Since this holds for each relator u_i, and since each letter in X is involved in some such relator, every letter of X (i.e. every generator of P) must lie in the group of units of P, so P is a group. ■

Remark We remark at this point that the word problem for one-relator monoids with a presentation of the form $M = Mon\langle X : u = 1\rangle$ was solved by Adian [1]. However the word problem for semigroups with one defining relation of the form $S = Sgp\langle X : u = v\rangle$ where both u and v are non-empty words in X^* remains open, as far as I am aware. There has been considerable work done on the one-relator semigroup problem in general (see for example, the papers by Adian and Oganessian [3], Guba [54], Lallement [75], Watier [144], and Zhang [148]). Later in this paper, I will indicate how this problem is related to the membership problem for certain submonoids of one-relator groups.

Despite the difficulties in deciding embeddability of a semigroup in a group in general, there are many significant results in the literature that show that monoids (semigroups) with particular presentations may be embedded in the corresponding groups. Perhaps the first such general result along these lines was obtained by Adian [2].

Let P be a semigroup with presentation $P = Sgp\langle X : u_i = v_i, i = 1, \dots n\rangle$, where u_i, v_i are strictly positive (i.e. non-empty) words. The *left graph* for this presentation is the graph with set X of vertices and with an edge from x to y if there is a defining relation of the form $u_i = v_i$ where x is the first letter of u_i and y is the first letter of v_i. The *right graph* is defined dually. The semigroup P is called an *Adian semigroup* and the corresponding group $G = Gp\langle X : u_i = v_i\rangle$ is called an *Adian group* if both the left graph and the right graph are cycle-free (i.e. if both graphs are forests). Of course a presentation is regarded as cycle-free if it contains no defining relations.

Theorem 2 *(Adian [2]) Any Adian semigroup embeds in the corresponding Adian group.*

Remmers [118] gave a geometric proof of this using semigroup diagrams, and Stallings [130] gave another proof using a graph theoretic lemma. Sarkisian [124], apparently gave a proof of the decidability of the membership problem for an Adian semigroup P in the corresponding Adian group G, and used this to solve the word problem for Adian groups: unfortunately there appears to be a gap in the proof in [124]. Adian's results have been extended in different directions in the work of several authors (see, for example, the papers by Kashintsev [70], Guba [53], Krstic [74], and Kilgour [72], where various small cancellation conditions are used to study embeddability of semigroups in groups).

We remark that in general an Adian group G is not the group of quotients of the corresponding Adian semigroup P. For example, if we consider the presentation $P = Sgp\langle a, b : ab = b^2 a\rangle$, then P is an Adian semigroup whose associated Adian group is the Baumslag-Solitar group $G = BS(1, 2)$. Not all elements of G belong to PP^{-1}, for example $a^{-1}ba \notin PP^{-1}$. However, every element of G can be written as a product of two elements of PP^{-1} - see Stallings [131] for a discussion of this example. Stallings shows that if P is an Adian semigroup, then PP^{-1} is a quasi-pregroup for the corresponding Adian group G (that is, if $q_1, q_2, \ldots, q_n \in PP^{-1}$, $n > 1$ and $q_i q_{i+1} \notin PP^{-1}$ for all i, then $q_1 q_2 \ldots q_n \neq 1$ in G).

As a second large class of important examples of semigroups that are embeddable in groups, we turn to a brief discussion of braid groups and Artin groups. The braid monoid on $n + 1$ strings is the monoid P_n with presentation

$Mon\langle x_1, x_2, \ldots, x_n : x_i x_j = x_j x_i$ if $|i - j| > 1, x_i x_{i+1} x_i = x_{i+1} x_i x_{i+1}$ if $i = 1, \ldots, n - 1\rangle$.

The corresponding group with the same presentation as a group, is the braid group B_n on $n + 1$ strands. Braid groups have been the object of intensive study in the literature (see for example the influential book of Birman [21], and many subsequent papers dealing with braids and their connection to other areas of mathematics). Braid monoids play a prominent role in the theory of braid groups. Garside [46] showed that such monoids satisfy the Ore condition, in fact the principal right ideals form a lattice: for each $a, b \in B_n$, there exists $c \in B_n$ such that $aB_n \cap bB_n = cB_n$, and also B_n is cancellative. Thus, by Ore's theorem we have:

Theorem 3 *(Garside) For each n, the braid monoid P_n embeds in the braid group B_n, and B_n is the group of quotients of P_n. Furthermore, the principal right ideals of P_n form a semilattice (in fact a lattice) under intersection.*

This result was simultaneously generalized by Brieskorn and Saito [23] and by Deligne [41] to Artin groups and monoids of finite type. Recall that a group G is called an *Artin group* and the corresponding monoid is called an *Artin monoid* if it is presented by a set X subject to relations of the form $prod(x, y; m_{x,y}) = prod(y, x; m_{x,y})$ if $m_{x,y} < \infty$. (Here $m_{x,x} = 1$ and $m_{x,y} = m_{y,x} \in \{2, 3, \ldots, \infty\}$ for $x, y \in X$, and $prod(x, y; m_{x,y})$ stands for the alternating word $xyxy\ldots$ of length $m_{x,y}$). An Artin group (monoid) is said to be *of finite type* if the corresponding Coxeter group is finite. These results were further generalized by Dehornoy and Paris [37] to a class of groups known as *Garside groups*, and were generalized further by Dehornoy [34] to a class of groups that admit a *thin* group of fractions, and to a group that arises in the study of left self distributivity and its connection to mathematical logic (see the book by Dehornoy [35] for full details about this). Many properties of braid groups, Artin groups of finite type, Garside groups and the more general groups considered by Dehornoy are proved by a deep study of the associated monoid of positive elements. We refer to the papers of Dehornoy cited above for further references and details. These groups admit a presentation where every relation is of the form $xu = yv$ for $x \neq y \in X$ and admit one such relation for each pair $x \neq y \in X$, so their left graphs are in fact cliques. Thus this class of groups is very different from the class of Adian groups. We also refer to the recent papers by Paris [109] and Godelle and Paris [50] where the authors solve Birman's conjecture [22] for braid groups and right angled Artin groups by studying the emebedding of singular braid monoids (Artin monoids) in the corresponding groups.

Several authors have studied the question of embeddability of general Artin monoids in Artin groups: for example special cases of this question have been considered by Charney [25] and Cho and Pride [28]. Much additional information about embeddability of semigroups in groups may be found in the paper by Cho and Pride. Paris [108] has established the following deep general result about Artin groups and Artin monoids.

Theorem 4 *(Paris) Every Artin monoid embeds in the corresponding Artin group.*

It is worth remarking that while Artin groups of finite type and the more general groups considered by Dehornoy et al are groups of fractions of their corresponding monoids of positive elements, this is not the case for Artin groups in general. The fact that braid groups, Artin groups of finite type, thin groups of fractions etc are all groups of fractions of their positive monoids leads to a fast algorithm for solving the word problem for such groups - they have quadratic isoperimetric inequality and admit an automatic structure. However, the word problem for Artin groups in general remains open, as far as I am aware.

I will close this section with brief mention of another prominent example of a monoid that embeds in its group of fractions. Recall that the Thompson group F can be defined by the presentation $F = Gp\langle x_0, x_1, \ldots : x_n x_k = x_k x_{n+1} \text{ for } k < n\rangle$.

This group has appeared in numerous settings, having been originally introduced by R. Thompson (see [98]) as a group that acts naturally on bracketed expressions by moving the brackets , i.e. by applying the associative law. We refer the reader to the monograph by Cannon, Floyd and Parry [24] for an introduction to the Thompson group F and some of its many connections with other areas of mathematics. The Thompson monoid is the monoid defined by the same relations as those that define F as a group. The following result appears to be well known.

Theorem 5 *The Thompson monoid embeds in the Thompson group F. Furthermore, F is the group of fractions of the Thompson monoid.*

A closely related group is the group G_{LD} introduced by Dehornoy [36] to describe the geometry of the left self-distributive law $x(yz) = (xy)(xz)$. See Dehornoy's book [35] for further information and deep connections with mathematical logic. It is known that the group G_{LD} is the group of quotients of an appropriate submonoid of this group, but a presentation for that submonoid seems to be unknown.

There are several ways to associate an **inverse monoid** with the situation when P is a monoid that embeds in a group G with the same presentation. We recall first some basic definitions and facts about regular and inverse monoids, and some of the rather extensive structure theory for such monoids.

3 Regular and Inverse Monoids

A monoid M is called a *(von Neumann) regular* monoid, if for each $a \in M$ there exists some element $b \in M$ such that $a = aba$ and $b = bab$. Such an element b is called *an inverse* of a (it is not necessarily unique). Note that regular monoids have in general lots of *idempotents*: if b is an inverse of a in M then ab and ba are both idempotents of M, i.e. $(ab)^2 = ab$ and $(ba)^2 = ba$ (and in general $ab \neq ba$). A monoid M is called an *inverse monoid* if for each $a \in M$ there exists a unique inverse (denoted by a^{-1}) in M such that

$$a = aa^{-1}a \quad \text{and} \quad a^{-1} = a^{-1}aa^{-1}.$$

Equivalently, M is inverse iff it is regular and the idempotents of M commute. Thus if M is an *inverse* monoid then the idempotents of M form a commutative idempotent semigroup with respect to the product in M. Since a commutative idempotent semigroup may be viewed as a lower semilattice (with meet operation equal to the product), we normally refer to such semigroups as *semilattices*. We will consistently denote the set of idempotents of a monoid M by $E(M)$. Thus if M is an inverse monoid, then $E(M)$ is a submonoid of M that is a semilattice, referred to as the *semilattice of idempotents* of M. Every inverse monoid M comes equipped with a *natural partial order* defined by

$$a \leq b \text{ if and only if } a = eb \text{ for some idempotent } e \in M.$$

If $e = e^2$ is an idempotent of a monoid M, then the set

$$H_e = \{a \in M : ae = ea = e \text{ and } \exists b \in M \text{ such that } ab = ba = e\}$$

is a *subgroup* of M with identity e (i.e. it forms a group with identity e relative to the multiplication in M). Clearly H_e is the largest subgroup of M with identity e, and $H_e \cap H_f = \emptyset$ if $e \neq f$. It is also clear that $H_1 = U(M)$, the group of units of the monoid M. The subgroups $H_e, e \in E(M)$ are referred to as the *maximal subgroups* of M. The semilattice of idempotents and the maximal subgroups of an inverse monoid M give us a good deal of information about M, but do not by any means determine the structure of M: in general, not all elements of an inverse monoid need belong to subgroups of the monoid.

A standard example of a regular monoid is the *full transformation monoid* on a set X, which consists of all functions from X to itself with respect to composition of functions. The group of units of this monoid is of course the symmetric group on X. Idempotents in this monoid consist of functions that are identity maps on their ranges, and the maximal subgroup corresponding to such an idempotent is isomorphic to the symmetric group on the range of the map. Every semigroup can be embedded in an appropriate full transformation monoid (see Clifford and Preston [30], Volume 1).

Another standard example of a regular semigroup is the *full linear monoid* $M_n(k)$ of $n \times n$ matrices with entries in a field k, with respect to matrix multiplication. The group of units of $M_n(k)$ is the general linear group $GL_n(k)$. From elementary linear algebra we know that an idempotent matrix of rank r is similar to the diagonal matrix with block diagonal identity matrix I_r in the top left hand corner and zeroes elsewhere. The group $GL_n(k)$ acts by conjugation on the set of idempotent matrices, and the orbits of this action consist of idempotent matrices of a fixed rank. The idempotent matrices in $M_n(k)$ may be identified with pairs of opposite parabolic subgroups of $GL_n(k)$. The maximal subgroup corresponding to an idempotent matrix of rank r is isomorphic to the general linear group $GL_r(k)$. Of course the idempotents of $M_n(k)$ do not form a subsemigroup if $n > 1$. We refer to Okninski's book [106] for a detailed description of this monoid, and to Putcha's book [114] for an introduction to the elegant theory of linear algebraic monoids. A linear algebraic monoid is regular if and only if its group of units is a reductive group: the subgroup structure of the group of units of a linear algebraic monoid provides very detailed information about the structure of the monoid (see [114]).

Clearly every group is an inverse monoid (in fact groups are just regular monoids with precisely one idempotent), and every semilattice E is an inverse monoid with $e^{-1} = e$ and with $H_e = \{e\}$, for all $e \in E$. A more enlightening example of an inverse monoid is the *symmetric inverse monoid* on a set X, denoted by $SIM(X)$. The monoid $SIM(X)$ is the monoid of all partial one-one maps (i.e. one-one maps from subsets of X to subsets of X) with respect to multiplication of partial maps: if α and β are partial one-one maps, then $\alpha\beta(x) = \alpha(\beta(x))$ whenever this makes sense, i.e. if $x \in dom(\beta)$ and $\beta(x) \in dom(\alpha)$. The group of units of $SIM(X)$ is obviously the symmetric group (the group of permutations) on X, and the idempotents of $SIM(X)$ are the identity maps on subsets of X, so the semilattice of idempotents of $SIM(X)$ is isomorphic to the lattice of subsets of X. The empty subset corresponds to the zero of $SIM(X)$: a product $\alpha\beta$ of two partial one-one maps on X is zero (the empty map) if $range(\beta) \cap dom(\alpha) = \emptyset$. The maximal subgroup corresponding to the identity map on the subset Y of X is the symmetric group on Y. The natural partial order on $SIM(X)$ is defined by domain restriction of a partial one-one map, i.e. $\alpha \leq \beta$ iff $dom(\alpha) \subseteq dom(\beta)$ and $\alpha = \beta|_{dom(\alpha)}$. (I note that the definition of $SIM(X)$ given here is the dual of the usual definition found in many books on semigroup theory, where functions are traditionally written on the right rather than the left.)

Symmetric inverse monoids are in a sense generic inverse monoids.

Theorem 6 *(Vagner-Preston) Every inverse monoid embeds in a suitable symmetric inverse monoid.*

Thus inverse monoids may be viewed as monoids of partial one-one maps, in much the same way as groups may be viewed as groups of permutations. Inverse monoids arise naturally whenever one encounters partial one-one maps throughout mathematics. For example, the Vagner-Preston theorem has been extended by Barnes [10] and Duncan and Paterson [44] to show that every inverse monoid embeds as a monoid of partial isometries of some Hilbert space, and from this point of view, inverse monoids play an increasingly important role in the theory of operator algebras (see the book by Paterson [110] for an introduction to the role of inverse monoids in this theory). The book by Petrich [111] or the more recent book by Lawson [76] provide an account of the general theory of inverse monoids and some of their connections with other areas of mathematics.

Another natural class of examples of inverse monoids arises in connection with submonoids of groups. Note that any submonoid P of a group must be a left and right cancellative monoid. Let P be any left cancellative monoid. The left regular representation $a \to \lambda_a$, where $\lambda_a : x \to ax$ for all $a, x \in P$, defines an embedding of P into the symmetric inverse monoid $SIM(P)$, since each map λ_a is clearly a partial one-one map on P with domain P and range aP. The submonoid of $SIM(P)$ generated by the image of P in this embedding into $SIM(P)$ is an inverse monoid, referred to as the *(left) inverse hull* $I_l(P)$ of P. Of course there is a dual inverse monoid $I_r(P)$ that arises from the right regular representation of a right cancellative monoid P. This is the most obvious way in which inverse monoids arise in connection with submonoids of groups. I will discuss some other ways of associating inverse monoids with submonoids of groups later in this paper.

The ideal structure of a monoid provides a basic tool for beginning to study the structure of the monoid. It will be convenient to introduce some standard terminology along these lines. There are five equivalence relations, known as the *Green's relations* $\mathcal{R}, \mathcal{L}, \mathcal{J}, \mathcal{H}$ and $\mathcal{D}$ that play a prominent role in the theory. For a monoid M we define

$$\begin{gathered}\mathcal{R} = \{(a,b) \in M \times M : aM = bM\},\\ \mathcal{L} = \{(a,b) \in M \times M : Ma = Mb\},\\ \mathcal{J} = \{(a,b) \in M \times M : MaM = MbM\},\\ \mathcal{H} = \mathcal{R} \cap \mathcal{L},\end{gathered}$$

and $\mathcal{D} = \mathcal{R} \vee \mathcal{L}$ (the join of $\mathcal{R}$ and $\mathcal{L}$ in the lattice of equivalence relations on M).

The corresponding equivalence classes containing $a \in M$ are denoted by R_a, L_a, J_a, H_a and D_a respectively. Clearly $\mathcal{H} \subseteq \mathcal{R}, \mathcal{L} \subseteq \mathcal{D} \subseteq \mathcal{J}$. It is a fortunate fact in semigroup theory that the equivalence relations $\mathcal{R}$ and $\mathcal{L}$ commute, i.e. $\mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$, and it follows that $\mathcal{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$. Thus $a\mathcal{D}b$ in M iff $\exists c \in M$ such that $a \,\mathcal{R}\, c \,\mathcal{L}\, b$ iff $\exists d \in M$ such that $a \,\mathcal{L}\, d \,\mathcal{R}\, b$. For an inverse monoid M it is easy to see that $a\,\mathcal{R}\,b$ iff $aa^{-1} = bb^{-1}$ and $a\,\mathcal{L}\,b$ iff $a^{-1}a = b^{-1}b$. It is a well-known fact that if P is a cancellative monoid, then P embeds as the $\mathcal{R}$-class R_1 of 1 in its right inverse hull $I_r(P)$ and as the $\mathcal{L}$-class L_1 of 1 in its left inverse hull $I_l(P)$.

It is informative to provide an explicit description of the Green's relations in the full linear monoid $M_n(k)$. If A and B are two matrices in $M_n(k)$, then

$A\,\mathcal{R}\,B$ iff $A\,GL_n(k) = B\,GL_n(k)$ iff $Col(A) = Col(B)$,
$A\,\mathcal{L}\,B$ iff $GL_n(k)\,A = GL_n(k)\,B$ iff $Nul(A) = Nul(B)$,
$A\,\mathcal{J}\,B$ iff $GL_n(k)\,A\,GL_n(k) = GL_n(k)\,B\,GL_n(k)$ iff $rank(A) = rank(B)$, and
$\mathcal{J} = \mathcal{D}$.

Furthermore, for each fixed $r \leq n$, the group $GL_n(k)$ acts transitively by left multiplication [resp. right multiplication] on the set of $\mathcal{R}$-classes of $M_n(k)$ [resp. $\mathcal{L}$-classes of $M_n(k)$] within the $\mathcal{J}$-class J_r consisting of the matrices of rank r. In addition, If Y_r denotes the set of all matrices of rank r that are in reduced row echelon form and if X_r is the set of transposes of elements of Y_r, then the $\mathcal{R}$-classes of $M_n(k)$ in the $\mathcal{J}$-class J_r are in one-one correspondence with the matrices in X_r, and the $\mathcal{L}$-classes of $M_n(k)$ in the $\mathcal{J}$-class J_r are in one-one correspondence with the matrices in Y_r. Every matrix in J_r has a unique decomposition of the form XGY with $X \in X_r, Y \in Y_r$ and $G \in GL_n(k)$. Proofs of all of these facts and much additional interesting information about full linear monoids may be found in [106].

It is a well known fact in semigroup theory that if a $\mathcal{D}$-class contains a regular element, then every element of that $\mathcal{D}$-class is regular. The structure of regular $\mathcal{D}$-classes is very nice: for example, all $\mathcal{H}$-classes within the $\mathcal{D}$-class are of the same cardinality, an $\mathcal{H}$-class is a maximal subgroup iff it contains an idempotent, and two maximal subgroups contained in the same $\mathcal{D}$-class are isomorphic. A $\mathcal{D}$-class is regular iff it contains an idempotent, and if a is a regular element of M, then every inverse of a lies in the $\mathcal{D}$-class D_a. A product ab lies in the $\mathcal{H}$-class $R_a \cap L_b$ iff $L_a \cap R_b$ contains an idempotent. Proofs of these facts may be found in any standard book on semigroup theory, for example [30].

A semigroup S is called *simple* [resp. *bisimple*] if it contains just one $\mathcal{J}$-class [resp. $\mathcal{D}$-class]. A semigroup with just one $\mathcal{H}$-class is a group. A semigroup S with a zero element 0 is called 0-*simple* if $S^2 \neq 0$ and S has only two $\mathcal{J}$-classes ($\{0\}$ and $S - \{0\}$). Such a semigroup is called 0-*bisimple* if it has just two $\mathcal{D}$-classes ($\{0\}$ and $S - \{0\}$). The structure of finite simple and 0-simple semigroups was determined by Suschkewitsch in 1928 [142]. This was extended by Rees [117] in 1940 to a class of simple [resp 0-simple] semigroups known as *completely simple* [resp. *completely* 0-*simple*] semigroups. We refer to [30] for an account of this important work.

Bisimple inverse monoids may be constructed from right cancellative monoids whose principal left ideals form a semilattice. The following theorem was proved by Clifford [29] in 1953.

Theorem 7 *(Clifford, 1953) Let M be a bisimple inverse monoid with identity 1 and let $R = R_1$, the $\mathcal{R}$-class of 1. Then R is a right cancellative monoid and the principal left ideals of R form a semilattice under intersection, i.e. for each $a, b \in R$, there exists $c \in R$ such that $Ra \cap Rb = Rc$. Conversely, let R be a right cancellative monoid in which the intersection of any two principal left ideals is a principal left ideal. Then the (right) inverse hull of R is a bisimple inverse monoid and the $\mathcal{R}$-class of 1 in this monoid is a submonoid that is isomorphic to R.*

Again, there is an obvious dual construction of bisimple inverse monoids from left cancellative monoids whose principal right ideals form a semilattice. We thus have the following corollary of Garside's theorem (Theorem 3) and Clifford's theorem (Theorem 7): the result extends to Artin groups of finite type, Garside groups, thin groups of quotients etc.

Corollary 1 *The inverse hull of the braid monoid B_n is a bisimple inverse monoid.*

If S is an inverse semigroup, then the natural partial order induces a homomorphism from S onto its maximal group homomorphic image. For S an inverse semigroup and $a, b \in S$ we define an equivalence relation σ on S by $a \, \sigma \, b$ iff $\exists c \in S$ such that $c \leq a$ and $c \leq b$. It is easy to see that σ is a *congruence* on S (i.e. it is compatible with respect to multiplication on both sides), so the set of σ-classes of S forms a semigroup S/σ with respect to the obvious multiplication, and there is a natural map (which we denote again by σ) from S onto S/σ. It is straightforward to see that S/σ is a group, the maximal group homomorphic image of S.

The inverse semigroup S is called *E-unitary* if the inverse image under σ of the identity of the group S/σ consists just of the semilattice $E(S)$ of idempotents of S. Equivalently, S is E-unitary if $a \geq e$ and $e \in E(S)$ implies $a \in E(S)$. E-unitary inverse semigroups play an essential role in the theory of inverse semigroups. Their structure has been determined by McAlister [93] by means of a group acting by order automorphisms on a partially ordered set with an embedded semilattice. Furthermore, McAlister proved [94] that if S is any inverse semigroup, then there is some E-unitary inverse semigroup T and an *idempotent-separating* homomorphism f from T onto S (a homomorphism $f : T \to S$ is called "idempotent-separating" if distinct idempotents of T are mapped to distinct idempotents of S). In this situation, we refer to T as an *E-unitary cover* of S over the group G, where G is the maximal group homomorphic image of T. We refer to Lawson [76] for a proof of McAlister's results and for an account of the importance of E-unitary inverse semigroups in the theory.

An inverse monoid M is called an *F-inverse monoid* if each σ-class of M contains a unique maximal element in the natural partial order on M. An F-inverse monoid is E-unitary. It is known that every inverse semigroup has an F-inverse cover, but it appears to be unknown at present whether every *finite* inverse semigroup has a *finite* F-inverse cover. One can define a multiplication on the maximal elements in σ-classes of an F-inverse monoid M as follows: if a and b are maximal elements in M (i.e. maximal elements in their respective σ-classes), then define $a.b$ to be the maximal element in the σ-class containing ab. With respect to this multiplication, the set of maximal elements of an F-inverse monoid forms a group that is isomorphic to the maximal group homomorphic image of M.

As an example of this situation, we recall some of the ideas developed by Birget [18] to study the complexity of the word problem for the Thompson group V. A subset R of a monoid M is called a right ideal if $RM \subseteq R$. A function $\theta : R \to R'$ is called a right ideal isomorphism if R and R' are right ideals of M and $\theta(rm) = \theta(r)m$ for all $r \in R, m \in M$. The collection of right ideal isomorphisms is an inverse monoid. A right ideal R of M is called an *essential right ideal* if $R \cap R' \neq \emptyset$ for all right ideals R'. The collection of right ideal isomorphisms between essential right ideals is an inverse monoid. Every essential right ideal of M is of the form $R = CM$ where C is a maximal prefix code, and every finitely generated essential right ideal of M is of the form $R = CM$ where C is a maximal finite prefix code. The collection of right ideal isomorphisms between the essential finitely generated right ideals of the free monoid $\{a, b\}^*$ on 2 letters is an F-inverse monoid whose maximal group image is Thompson's group V: the multiplication that defines the group V is just the multiplication of maximal elements in σ-classes defined above (see Birget's paper [18] for details). The tree representation of prefix codes connects this definition and the definition by action on finite trees used in [24].

There are many other examples of this situation in the literature. I will briefly mention the work of Lawson [77] on *Möbius inverse monoids*. Recall that a *Möbius transformation* is a linear fractional transformation α of the complex plane having the form $\alpha(z) = (az+b)/(cz+d)$ where a, b, c, d are complex numbers and $ad - bc \neq 0$. Möbius transformations are either one-one functions (when $c = 0$) or partial one-one functions with restricted domain (when $c \neq 0$). Thus they are elements of the symmetric inverse monoid $SIM(\mathbf{C})$. The *Möbius inverse monoid* is the (inverse) submonoid of $SIM(\mathbf{C})$ generated by the Möbius transformations. It is an F-inverse monoid with maximal group image the *Möbius group*. One may refer to [77] or Lawson's book [76] for details.

An E-unitary inverse monoid with a zero must be essentially trivial - more precisely, it must be a semilattice with 0 and 1. An inverse semigroup S with 0 is called E^*-unitary if, whenever $a \geq e = e^2$ and $e \neq 0$ in S, then $a = a^2$. Similarly, an inverse monoid M is called F^*-inverse if each non-zero element of M has a unique maximal element in its σ-class. The multiplication on the maximal elements of each σ-class that was defined above for F-inverse monoids essentially carries over to the case of F^*-inverse monoids. The product of two such maximal elements $a.b$ is defined as before if $ab \neq 0$ and it is undefined if $ab = 0$. It is of interest to know when the partial group of maximal elements of such a monoid embeds in a group: this happens, for example, if this partial group is a pregroup in the sense of Stallings [132]. F-inverse monoids and F^*-inverse monoids arise naturally in connection with submonoids of groups.

Let us return now to the situation where $P = Mon\langle X : u_i = v_i\rangle$ embeds as the positive submonoid in the group $G = Gp\langle X : u_i = v_i\rangle$ with the same presentation. For each $g \in G$ denote by β_g the restriction of the left regular representation of G to P, i.e. $dom(\beta_g) = \{t \in P : gt \in P\}$, $range(\beta_g) = \{s \in P : g^{-1}s \in P\}$, and β_g acts by left translation by g on elements of its domain. Clearly $\beta_g \in SIM(P)$ for each $g \in G$. Let $T(G,P)$ be the inverse submonoid of $SIM(P)$ generated by $\{\beta_g : g \in G\}$. $T(G,P)$ is referred to as the *Toeplitz inverse monoid of* (G,P) because of its natural connection to the Toeplitz algebra associated with G and P. It was introduced in the paper of Nica [104] who constructed a locally compact space on which the monoid $T(G,P)$ acts and who showed that this action leads very naturally to the *Wiener-Hopf groupoid* whose reduced C^*-algebra is the Toeplitz algebra associated with (G,P). The locally compact space on which $T(G,P)$ acts may be viewed essentially as a compactifiction of the set of $\mathcal{R}$-classes in the $\mathcal{D}$-class of 1 in $T(G,P)$ (see the paper by Nica [104] for details about this).

For example, if $P = \mathbf{N}$ and $G = \mathbf{Z}$, then $T(G,P)$ is the *bicyclic monoid*, often defined by the monoid presentation $Mon\langle a,b : ab = 1\rangle$. The bicyclic monoid is a bisimple F-inverse inverse monoid with an infinite descending chain of idempotents $1 > ba > b^2a^2 > b^3a^3 > \ldots$. If $P = X^*$ and $G = FG(X)$ (the free group on X), then $T(G,P)$ is the *polycyclic monoid on* X, which is defined by the monoid presentation $Mon\langle X \cup X^{-1} : x^{-1}x = 1, x^{-1}y = 0 \text{ for } x \neq y\rangle$. The polycyclic monoid was first introduced by Nivat and Perrot [105] as the syntactic monoid of the language of correctly parenthesized expressions. It is F^*-inverse and 0-bisimple. It has also surfaced in operator algebras in connection with Cuntz $\mathbf{C}^*$-algebras, where it is referred to as the *Cuntz monoid* (see [110]) and in many other contexts, including Birget's work on Thompson's group V [18]. For the polycyclic monoid, the space constructed by Nica on which this monoid acts is a subspace of the space of ends of the tree of the free monoid X^*.

The following result follows from the work of Nica [104].

Theorem 8 *Suppose that $P = Mon\langle X : u_i = v_i\rangle$ embeds as the positive submonoid in the group $G = Gp\langle X : u_i = v_i\rangle$ with the same presentation. Then*

(1) $T(G,P)$ contains a zero iff $PP^{-1} \neq G$.

(2) $T(G,P)$ is a simple F-inverse monoid if $G = PP^{-1}$ and a 0-simple F^*-inverse monoid if $G \neq PP^{-1}$.

Additional information about the structure of $T(G,P)$ and its connection with the inverse hull $I_l(P)$ has been developed by Margolis and Lawson (private communication). For braid groups, Artin groups of finite type, Garside groups etc, the corresponding Toeplitz inverse monoid is bisimple, and is equal to the (left) inverse hull of the monoid, but for general Artin groups, Adian groups etc, the corresponding Toeplitz inverse monoid is F^* inverse and 0-simple.

Inverse monoids arise in many other ways in the context of geometric group theory. In order to describe some of these connections, it will be convenient to recall some of the beautiful theory of free inverse monoids.

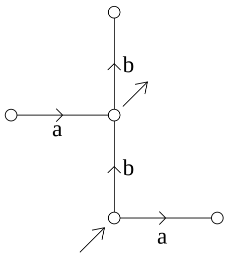
4 Free Inverse Monoids, Equations

Inverse monoids form a variety of algebras in the sense of universal algebra, defined by associativity and the identities:

$$a = aa^{-1}a,\ (a^{-1})^{-1} = a,\ (ab)^{-1} = b^{-1}a^{-1},\ aa^{-1}bb^{-1} = bb^{-1}aa^{-1}.$$

It follows that free inverse monoids exist. We will denote the free inverse monoid on a set X by $FIM(X)$. Clearly $FIM(X)$ is the quotient of the free monoid $(X \cup X^{-1})^*$ by the smallest congruence that makes the identities above hold. An elegant solution to the word problem for $FIM(X)$ was obtained by Munn [100] in 1974.

Denote by $FG(X)$ the free group on X and by $\Gamma(X)$ the Cayley graph of $(FG(X), \emptyset)$. For each word $w \in (X \cup X^{-1})^*$, denote by $MT(w)$ the finite subtree of the tree $\Gamma(X)$ obtained by reading the word w as the label of a path in $\Gamma(X)$, starting at 1. Thus, for example, if $w = aa^{-1}bb^{-1}ba^{-1}abb^{-1}$, then $MT(w)$ is the tree pictured below.



One may view $MT(w)$ as a birooted tree, with initial root 1 and terminal root $r(w)$, the reduced form of the word w in the usual group-theoretic sense. Munn's solution to the word problem in $FIM(X)$ may be stated in the following form.

Theorem 9 *(Munn) If $u, v \in (X \cup X^{-1})^*$, then $u = v$ in $FIM(X)$ iff $MT(u) = MT(v)$ and $r(u) = r(v)$.*

Thus elements of $FIM(X)$ may be viewed as pairs $(MT(w), r(w))$ (or as birooted edge-labelled trees, which was the way that Munn described his results). An equivalent description of $FIM(X)$ in terms of Schreier subsets of $FG(X)$ was provided independently by Scheiblich [125]. Multiplication in $FIM(X)$ is performed as follows. If $u, v \in (X \cup X^{-1})^*$, then $MT(uv) = MT(u) \cup r(u).MT(v)$ (just translate $MT(v)$ so that its initial root coincides with the terminal root of $MT(u)$ and take the union of $MT(u)$ and the translated copy of $MT(v)$: the terminal root is of course $r(uv)$).

It is straightforward to see that $FIM(X)$ is an F-inverse monoid with maximal group image $FG(X)$: the natural map $\sigma : FIM(X) \to FG(X)$ takes $(MT(u), r(u))$ onto $r(u)$. The maximum element in a σ-class is just the the reduced word in that σ-class. The multiplication of maximal elements in the F-inverse monoid $FIM(X)$ is just the well-known multiplication in the free group: $r(u).r(v) = r(r(u)r(v))$. The idempotents of $FIM(X)$ correspond to the Dyck words in $(X \cup X^{-1})^*$, i.e. the words whose reduced form is 1. Two such words represent the same idempotent in $FIM(X)$ iff they span the same Munn tree. The maximal subgroups of $FIM(X)$ are all trivial (i.e. $FIM(X)$ is *combinatorial*). Free inverse monoids are residually finite and the word problem is decidable in linear time by Munn's theorem.

The construction of $FIM(X)$ from finite birooted subtrees of $\Gamma(X)$ discussed above can be extended to the Cayley graph of any group presentation. Let $\Gamma(G, X)$ be the Cayley graph of a group G relative to a set X of generators. Let $M(G, X) = \{(\Delta, g) : \Delta$ is a finite connected subgraph of $\Gamma(G, X)$ containing 1 and $g\}$, with multiplication $(\Delta_1, g_1).(\Delta_2, g_2) = (\Delta_1 \cup g_1.\Delta_2, g_1 g_2)$. Then $M(G, X)$ is an X-generated E-unitary inverse monoid with maximal group image G, in fact it is the "universal" X-generated E-unitary inverse monoid with maximal group image G (see the paper by Margolis and Meakin [85] for details about this construction).

There are some similarities and some striking differences between the theory of free groups and the theory of free inverse monoids. It is easy to see that inverse submonoids of free inverse monoids are not necessarily free: for example, the set of idempotents of a free inverse monoid is an inverse submonoid that is a semilattice, so it is certainly not a free inverse monoid. However, the study of *closed* inverse submonoids of free inverse monoids has some interesting relationships with the study of subgroups of free groups: this will be described in the next section of this paper. There is a developing theory of presentations of inverse monoids by generators and relations: some results about inverse monoid presentations will be discussed in the final section of this paper. The theory of equations in free inverse monoids is significantly different than the corresponding theory for free groups or free monoids. We describe some aspects of this situation in the remainder of this section.

Let X be an alphabet that is disjoint from A. It is convenient to denote $X \cup X^{-1}$ by $\tilde{X}$ and $A \cup A^{-1}$ by $\tilde{A}$ throughout the remainder of this section. We will view letters of $\tilde{X}$ as *variables* and elements of $\tilde{A}^*$ as *constants*. The sets A and X will be assumed to be *finite and non-empty*. An *equation* in $FG(A)$ or in $FIM(A)$ with coefficients in $FG(A)$ (or in $FIM(A)$) is a pair (u, v), where $u, v \in (\tilde{A} \cup \tilde{X})^*$. Usually we will denote such an equation by $u = v$: if necessary for emphasis we will denote u and v by $u(X, A)$ and $v(X, A)$ if there is any possibility of confusion about the sets of variables and constants in the equation. Similarly an equation in A^* is a pair (u, v) with $u, v \in (A \cup X)^*$, and again we will denote this by $u = v$. If needed to distinguish where equations are being viewed, we will denote an equation $u = v$ in A^*, [resp. $FG(A)$, $FIM(A)$] by $u =_M v$ [resp. $u =_G v, u =_I v$].

Any map $\phi : X \to \tilde{A}^*$ extends to a homomorphism (again denoted by ϕ) from $(\tilde{A} \cup \tilde{X})^*$ to $\tilde{A}^*$ in such a way that ϕ fixes the letters of A. We say that ϕ is a *solution* to the equation $u =_G v$ in $FG(A)$ [resp. $u =_I v$ in $FIM(A)$ or $u =_M v$ in A^*] if $\phi(u) = \phi(v)$ in the appropriate setting. A solution to a set of equations $u_i = v_i$ for $i = 1, \ldots n$ is a map ϕ that is a solution to each equation in the set. If a set of equations has at least one solution it is called *consistent*: otherwise it is called *inconsistent*. It is easy to give examples of equations that are inconsistent in any of the three possible settings where we are considering such equations, and it is easy to give examples of equations that are consistent in $FG(A)$ but not in $FIM(A)$ or in A^*. For example, if $A = \{a, b\}$, then the equation $ax = xb$ is inconsistent in all three settings, while the equation $ax = b$ is consistent in $FG(A)$ but inconsistent in A^* and in $FIM(A)$ if $a \neq b$. [It is obvious that this equation is inconsistent in A^* - no word in a free monoid can start with two distinct letters: one sees easily that this equation is also inconsistent in $FIM(a, b)$ - no matter what x is, when one multiplies $MT(a)$ by $MT(x)$, there must be an edge labelled by a in the resulting tree, so this cannot be the Munn tree of b.]

On the other hand, it is obvious that any set of equations that is consistent in $FIM(A)$ must be consistent in $FG(A)$: if ψ is any solution to a set of equations in $FIM(A)$ and $\psi(x) = w_x \in \tilde{A}^*$ for each $x \in X$, then $\phi : X \to \tilde{A}^*$ defined by $\phi(x) = r(w_x)$ is a solution to the same set of equations, viewed as equations in $FG(A)$.

The *consistency problem* for systems of equations in A^* [resp. $FG(A), FIM(A)$] is the problem of determining whether there is an algorithm that, on input a finite set $\{u_i = v_i : i = 1 \dots n\}$ of equations in A^* [resp. $FG(A), FIM(A)$], produces an output of "Yes" if the system is consistent and "No" if it is inconsistent. Theorems of Makanin [83],[84] imply that the consistency problems for systems of equations in A^* and in $FG(A)$ are decidable. Much work has been done on solutions to systems of equations in free monoids and free groups: we refer the reader to [80, 52, 113, 116, 32] for just some of the extensive literature on this subject. On the other hand, a theorem of Rozenblat [121] shows that while the consistency problem for systems of equations in $FIM(A)$ is decidable if $|A| = 1$, this problem is undecidable if $|A| > 1$. The consistency problem for equations in $FIM(A)$ of some restricted type (for example, single variable equations, or quadratic equations) is open as far as I am aware. Some work on special cases of this problem has been done by Deis [38]. For example, Deis [38] has shown that while the consistency problem for single *multilinear* equations in $FIM(A)$ is decidable, the consistency problem for finite *systems* of multilinear equations is undecidable.

Now consider an equation $u =_I v$ in $FIM(A)$, let ψ be a solution to this in $FIM(A)$, and let ϕ be a solution to the corresponding equation in $FG(A)$, where $\phi(x)$ is a reduced word for each $x \in X$. We say that ψ is an *extension* of ϕ (or that ϕ *extends to* ψ) if for each $x \in X$ there is some Dyck word e_x such that $\psi(x) = e_x\phi(x)$. If ψ is a solution to an equation $u = v$ in $FIM(A)$ and if $\phi(x) = r(\psi(x))$ for each $x \in X$, then of course ϕ is a solution to $u = v$ in $FG(A)$ and ψ is an extension of ϕ.

A given solution ϕ to an equation $u = v$ in $FG(A)$ may admit finitely many extended solutions, infinitely many extended solutions, or no extended solutions, to the same equation in $FIM(A)$. For example, the equation $bb^{-1}x = aa^{-1}bb^{-1}$ has trivial solution in $FG(a, b)$, and this has exactly two extensions $\psi_1(x) = aa^{-1}bb^{-1}$ and $\psi_2(x) = aa^{-1}$ in $FIM(a, b)$. The equation $bb^{-1}x = aa^{-1}x$ has trivial solution in $FG(a, b)$ that extends to infinitely many solutions $\psi_e(x) = e$ for any idempotent $e \leq aa^{-1}bb^{-1}$ in the natural order on $FIM(a, b)$. The equation $a^{-1}ax = aa^{-1}$ has trivial solution in $FG(a, b)$ but no solution in $FIM(a, b)$. These facts are easy to check via the multiplication of Munn trees in the free inverse monoid, as described above.

A natural question arises here: when does a solution to an equation $u = v$ in $FG(A)$ extend to a solution to the same equation in $FIM(A)$? We refer to the corresponding algorithmic problem as the *extendibility problem* for equations in $FIM(A)$. More precisely, the extendibility problem for equations in $FIM(A)$ asks whether there is an algorithm that, on input a finite set $\{u_i = v_i : i = 1, \dots n\}$ of equations in $FIM(A)$ that is consistent in $FG(A)$ and a solution ϕ to this system in $FG(A)$, produces the output "Yes" if ϕ can be extended to a solution to the system of equations in $FIM(A)$ and "No" if ϕ cannot be extended to a solution to this system in $FIM(A)$. Some special cases of the extendibility problem were considered by Deis [38]. The main result of the paper by Deis, Meakin and Sénizergues [39] shows that the extendibility problem is decidable.

In [39] it is shown that the requirement that ϕ should be extendible to some solution $\psi(x) = e_x\phi(x)$ to the system in $FIM(A)$ translates as follows. Consider the system of equations

$$\sum_x \alpha_{i,x} \cdot T_x + \beta_i = \sum_x \alpha'_{i,x} \cdot T_x + \beta'_i : i = 1, \dots, n \tag{1}$$

Here the $\alpha_{i,x}, \alpha'_{i,x}, \beta_i, \beta'_i$ are finite subsets of $FG(A)$ and the T_x are unknowns. A solution of (1) is any collection of subsets $T_x (x \in X)$ of $FG(A)$ that satisfies this system of equations. We would like to decide whether the system of equations (1) has at least one solution such that each T_x is both finite and prefix closed. (A subset T of $FG(A)$ is *prefix closed* if the corresponding set of reduced words is prefix closed.) In [39] it is shown that this problem is decidable by appealing to Rabin's tree theorem [115]. From the discussion above, this shows that the extendibility problem is decidable. Rabin's tree theorem is a theorem in second order monadic logic.

We assume some familiarity with basic definitions and ideas of (first order) logic. See, for example, Barwise [12]. In second order monadic logic, quantifiers refer to sets (i.e. unary or monadic predicates) as well as to individual members of a structure. The syntax and semantics of terms and well formed formulae are defined inductively in the usual way. Atomic formulae include those of the form $t \in Y$ where t is a term and Y is a set variable. A sentence of the form $\forall Y \nu(Y)$ where Y is a set variable, in particular, is true in a structure M iff $\nu(Y)$ is (inductively) true in M for all subsets Y of the universe of M. If a sentence θ is true in a structure M we write $M \models \theta$ and we define $Th_2(M) = \{\theta : M \models \theta\}$. The (second order monadic) theory of M is *decidable* if there is an algorithm that tests whether a given sentence θ of the language of M is in $Th_2(M)$ or not.

Let A be a countable set and consider the structure $T_A = (A^*, \{r_a : a \in A\}, \leq)$. Here $r_a : A^* \to A^*$ is right multiplication by a, $xr_a = xa, \forall x \in A^*$ and $\leq$ is the prefix order $x \leq y$ iff $\exists u \in A^*(xu = y)$. The theory $Th_2(T_A)$ is called the theory of A-successor functions. For $|A| = 2$ this is often denoted by $S2S$, and sentences in $Th_2(T_A)$ can be reformulated as sentences in $S2S$. Rabin's tree theorem [115] is that $Th_2(T_A)$ is decidable. It is one of the most powerful decidability results known in model theory: the decidability of many other results can be reduced to $Th_2(T_A)$ (see, for example, [12]). A rather delicate application of Rabin's tree theorem enables a proof of the main theorem of Deis, Meakin and Sénizergues [39].

Theorem 10 *(Deis, Meakin, Sénizergues) There is an algorithm that will decide, on input a system of equations of the form (1), whether this system of equations has at least one solution $\{T_x : x \in X\}$ such that each T_x is a finite prefix-closed subset of $FG(A)$. Hence the extendability problem for equations in $FIM(A)$ is decidable.*

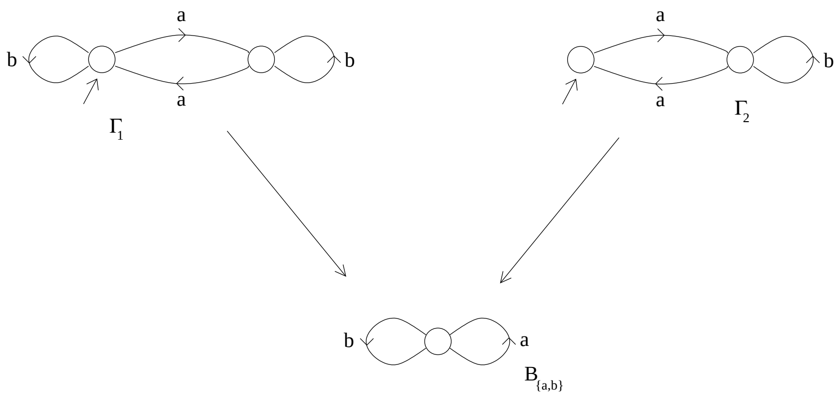
This result can be used to study the consistency problem for equations in $FIM(A)$ when one has good control over the set of solutions to the corresponding equations in $FG(A)$. For example, a result in [129] that is attributed to James Howie states that if $w(x) = 1$ is a single-variable equation in $FG(A)$ and the exponent sum of the single variable x in $w(x)$ is not zero, then the equation $w(x) = 1$ can have at most one solution in $FG(A)$. This implies that the consistency problem for such an equation in $FIM(A)$ is decidable, by Theorem 10. It is an open problem whether the consistency problem for all single variable equations in $FIM(A)$ is decidable. Some cases where this problem is shown to be decidable are contained in the paper [39]. In view of established literature providing parametric solutions to single variable equations in free groups (see Lyndon [81], Appel [5], Lorents [79]), it is plausible that the consistency problem for single variable equations in $FIM(A)$ might be decidable.

5 Subgroups of free groups and closed inverse submonoids of free inverse monoids

In his paper [133], Stallings showed how immersions between finite graphs (i.e locally injective graph morphisms) may be used to study finitely generated subgroups of free groups. In this section, we show that inverse monoids play the same role in the theory of immersions that groups play in the theory of coverings. This leads to the study of closed inverse submonoids of free inverse monoids and to the use of finite inverse monoids to classify finitely generated subgroups of free groups.

By a *graph* $\Gamma = (V(\Gamma), E(\Gamma))$ we will mean a graph in the sense of Serre [128]. Thus every edge $e : v \to w$ comes equipped with an inverse edge $e^{-1} : w \to v$ and $e^{-1} \neq e$: the initial vertex of e is denoted by $\alpha(e)$ and the terminal vertex of e is denoted by $\omega(e)$. There is an evident notion of morphism between graphs. If $v \in V(\Gamma)$, let $star(\Gamma, v) = \{e \in E(\Gamma) : \alpha(e) = v\}$: a morphism $f : \Gamma \to \Gamma'$ induces a map $f_v : star(\Gamma, v) \to star(\Gamma', f(v))$ between star sets. We say that f is a *cover* if each f_v is a bijection and that f is an *immersion* if each f_v is an injection. It is clear from the definition of graph that it is possible to label the edges of a graph with labels coming from a set $X \cup X^{-1}$ in such a way that if x labels an edge e, then x^{-1} labels e^{-1}, and so that the labelling is consistent with an immersion over B_X, the bouquet of $|X|$ circles: by this we mean that we choose a labelling set X such that no two edges with the same initial or terminal vertex are assigned the same label. The associated natural immersion $f_\Gamma : \Gamma \to B_X$ preserves edge labelling. *All graphs that we will consider in this paper will be so labelled and all immersions will preserve edge labelling.* For example, the Cayley graph $\Gamma(G, X)$ of a group G relative to a set X of generators is obviously labelled over $X \cup X^{-1}$. Munn trees are finite trees with edges labelled over $X \cup X^{-1}$ for some appropriate set X.

A simple example illustrating these ideas is provided in the diagram below: the natural map from the graph Γ_1 to $B_{\{a,b\}}$ is a cover, while the natural map from Γ_2 to $B_{\{a,b\}}$ is an immersion that is not a cover.



It is well known (see for example, Lyndon and Schupp [82], or Stallings [133] or Stillwell [134]) that covers of a connected graph Γ may be classified via subgroups of the fundamental group $\pi_1(\Gamma, v)$ based at some base point $v \in V(\Gamma)$. It is convenient for us for later use to adopt the slightly more general point of view of Higgins [60] and consider the fundamental groupoid of the graph Γ. Here the notion of homotopy on the set $P(\Gamma)$ of paths of Γ is defined from the equivalence relation $\sim$ on $P(\Gamma)$ induced by removing a pair of consecutive edges of the form ee^{-1} from a path p to get a path q. If we view $\sim$ as a congruence on the free category $C(\Gamma)$ over Γ, then the quotient category $\pi_1(\Gamma)$ is a *groupoid* (i.e. a small category in which each morphism is an isomorphism), called the *fundamental groupoid* of Γ. Denote the $\sim$-equivalence class containing the path p by $[p]$ and for each vertex $v \in V(\Gamma)$ let $\pi_1(\Gamma, v) = \{[p] \in \pi_1(\Gamma) : \alpha(p) = \omega(p) = v\}$. Then $\pi_1(\Gamma, v)$ is a group, called the *fundamental group of* Γ *based at* v. The fundamental groups are free groups, and fundamental groups at different base points in the same connected component of Γ are isomorphic. If Γ is connected and T is a spanning tree of Γ, then the rank of $\pi_1(\Gamma, v)$ is the number of positively oriented edges in $\Gamma - T$. For example, $\pi_1(B_X) \cong FG(X)$. We refer to standard sources such as Higgins [60] or Lyndon and Schupp [82] or Cohen [31] for the basic theory of coverings of a connected graph via subgroups of the fundamental group of the graph.

Closed inverse submonoids of free inverse monoids provide the analogous tool for classifying immersions between graphs. An inverse submonoid N of an inverse monoid M is called a *closed inverse submonoid of* M if, whenever $a \in N, b \in M$ and $b \geq a$ in the natural partial order on M, then $b \in N$. For example, by definition, an inverse monoid M is E-unitary iff $E(M)$ is a closed inverse submonoid of M. Closed inverse submonoids of an inverse monoid M arise naturally from transitive representations of M by partial one-one maps, in essentially the same way as subgroups of a group arise in connection with transitive representations of the group by permutations. We briefly review Schein's theory [126] of representations of inverse monoids by injective maps.

An inverse monoid M acts (on the left by injective functions) on a set Q if there is a morphism from M to $SIM(Q)$. If $m \in M$ and $q \in Q$ then we denote by mq the image of q under the action of m if $q \in Dom(m)$. An action is transitive if for all $p, q \in Q$ there exists $m \in M$ such that $mq = p$. Notice that this implies that $m^{-1}p = q$. For every $q \in Q, Stab(q) = \{m \in M : mq = q\}$ is a closed inverse submonoid of M. Conversely, given a closed inverse submonoid N of M, we can construct a transitive representation of M as follows. Let m be such that $m^{-1}m \in N$. A subset of M of the form $(mN)^{\omega} = \{s : s \geq mn$ for some $n \in N\}$ is called a *left ω-coset* of N. Notice that $N = (1N)^{\omega}$ is a left ω-coset of itself. Let X_N be the set of left ω-cosets of N. If $m \in M$, define an action on X_N by $m.Y = (mY)^{\omega}$ if $(mY)^{\omega} \in X_N$ and undefined otherwise. This defines a transitive action of M on X_N. Conversely, if M acts transitively on Q, then this action is equivalent to the action of M on the left ω-cosets of $Stab(q)$ for any $q \in Q$. For details we refer to Schein's paper [126] or Petrich's book [111]. Note that if M is a group, this just reduces to the usual coset representation of M on some subgroup N. Clearly there is a dual notion of right action and right ω-cosets of a closed inverse submonoid of M.

Now let Γ be a (connected) graph. In order to classify the immersions over Γ we make use of the free inverse category over Γ. A category C is called an *inverse category* (see [MM1]) if for each morphism p of C there is a unique morphism p^{-1} of C such that $p = pp^{-1}p$ and $p^{-1} = p^{-1}pp^{-1}$. Denote the loop monoid at a vertex (object) v of C by $Mor(v, v)$: that is, $Mor(v, v)$ is the set of morphisms p from v to v, with the multiplication induced by C. It is clear that each loop monoid is an inverse monoid. The *free inverse category* $FIC(\Gamma)$ over Γ is the quotient of the free category on Γ by the congruence $\sim_I$ induced by all relations of the form $p = pp^{-1}p$, $p^{-1} = p^{-1}pp^{-1}$ and $pp^{-1}qq^{-1} = qq^{-1}pp^{-1}$ when $\alpha(p) = \alpha(q)$ for paths p, q in Γ. Then $FIC(\Gamma)$ is an inverse category. If $\Gamma = B_X$, then $FIC(\Gamma) = FIM(X)$. The word problem in $FIC(\Gamma)$ may be solved in a manner similar to the way in which Munn solved the word problem for $FIM(X)$ by passing to the universal cover of Γ. If Γ is a connected graph labelled over $X \cup X^{-1}$ as described above, then each loop monoid of $FIC(\Gamma)$ is a closed inverse submonoid of $FIM(X)$. See the paper by Margolis and Meakin [86] for details and proofs of these facts.

Free inverse categories and their loop monoids serve the same role in classifying connected immersions as do fundamental groupoids and their corresponding fundamental groups in classifying connected covers of graphs. Let Γ be a graph labelled over $X \cup X^{-1}$ as above. Denote the loop monoid of $FIC(\Gamma)$ at vertex v by $L(\Gamma, v)$, a closed inverse submonoid of $FIM(X)$. If v and w are vertices in the same connected component of a graph Γ then $L(\Gamma, v)$ is *conjugate* to $L(\Gamma, w)$, that is, there exists $m \in FIM(X)$ such that $m^{-1}L(\Gamma, v)m \subseteq L(\Gamma, w)$ and $mL(\Gamma, w)m^{-1} \subseteq L(\Gamma, v)$. (This does not imply that the submonoids $L(\Gamma, v)$ and $L(\Gamma, w)$ are isomorphic however - see [86].) The following result is proved in [86].

Theorem 11 *(Margolis and Meakin) Let $f : \Delta \to \Gamma$ be an immersion over Γ, where Δ and Γ are connected graphs labelled over $X \cup X^{-1}$. If $v \in V(\Gamma)$ and $v' \in V(\Delta)$ such that $f(v') = v$,* *then f induces an embedding of $L(\Delta, v')$ into $L(\Gamma, v)$. Conversely, let Γ be a graph labelled over $X \cup X^{-1}$ and let H be a closed inverse submonoid of $FIM(X)$ such that $H \subseteq L(\Gamma, v)$ for some vertex $v \in V(\Gamma)$. Then there is a graph Δ, an immersion $f : \Delta \to \Gamma$ and a vertex $v' \in V(\Delta)$ such that $f(v') = v$ and $f(L(\Delta, v')) = H$. Furthermore, Δ is unique (up to graph isomorphism) and f is unique (up to equivalence). If H and K are two closed inverse submonoids of $FIM(X)$ with $H, K \subseteq L(\Gamma, v)$ then the corresponding immersions $f : \Delta \to \Gamma$ and $g : \Delta' \to \Gamma$ are equivalent if and only if H and K are conjugate in $FIM(X)$.*

While closed inverse submonoids of free inverse monoids are not necessarily free, they share many properties in common with free inverse monoids and they may be built from free actions of groups on trees and they admit idempotent-pure morphisms onto free inverse monoids ([86]).

Theorem 12 *(Margolis and Meakin) Let G be a group that acts freely on the left on a (simplicial) tree T (so that G is a free group). Fix a root $v_0 \in V(T)$. Let $M(T, G, v_0) = \{(t, g) : t$ is a finite subtree of T, $g \in G$, $v_0, g.v_0 \in V(t)\}$ with multiplication $(t_1, g_1).(t_2, g_2) = (t_1 \cup g_1.t_2, g_1g_2)$. Then $M(T, G, v_0)$ is isomorphic to a closed inverse submonoid of $FIM(X)$ where X may be identified with the positively oriented edges of the quotient graph of the action of G on T. Conversely, every closed inverse submonoid of a free inverse monoid arises this way. Furthermore, $M(T, G, v_0)$ admits an idempotent-pure morphism onto some free inverse monoid $FIM(Y)$ (i.e. there is a surjective morphism $f : M(T, G, v_0) \to FIM(Y)$ such that the inverse image of each idempotent in $FIM(Y)$ consists only of idempotents in $M(T, G, v_0)$).*

Closed inverse submonoids of free inverse monoids admit remarkable finiteness properties. We say that a closed inverse submonoid N of M is *finitely generated* (in the closed sense) if there is a finite set A of elements of M such that $N = \langle A \rangle^{\omega}$, where $\langle A \rangle$ denotes the inverse submonoid of M generated by A. We say that N *is of finite index in* M if there are finitely many left ω-cosets of N in M. A subset U of a monoid M is called *rational* if it can be built from the one-element subsets of M by finitely many applications of the Kleene operations of union, concatenation and submonoid generation: U is called *recognizable* if there is a finite monoid K and a subset $P \subseteq K$ and a morphism $f : M \to K$ such that $U = f^{-1}(P)$ (see Berstel [17]) for details). Rational and recognizable sets coincide in finitely generated free monoids by Klenne's theorem (see any book on formal language theory, e.g. Hopcroft and Ullman [61], for a proof of Kleene's theorem), but they do not coincide in general. A well known result of Anissimov and Seifert [6] shows that a subgroup H of a finitely generated group G is recognizable iff H has finite index in G and rational iff H is finitely generated. By way of contrast, for closed inverse submonoids of free inverse monoids, we have the following surprising result that is proved in [86].

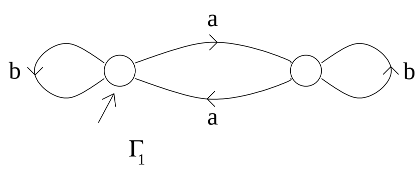
Theorem 13 *(Margolis and Meakin) Let N be a closed inverse submonoid of $FIM(X)$ for X a finite set. Then the following conditions are equivalent:*

(1) N has finite index in $FIM(X)$;
(2) N is recognizable;
(3) N is rational;
(4) N is finitely generated (in the closed sense).

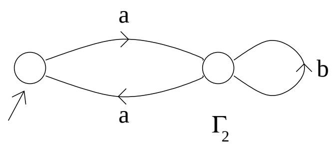
In his paper [133], Stallings shows how finite $X \cup X^{-1}$-labelled graphs (i.e. finite immersions over B_X) may be used to study finitely generated subgroups of free groups. We recall the basic construction of these graphs. Let H be a subgroup of $FG(X)$ generated by the reduced words $h_1, h_2, \ldots, h_n \in FG(X)$. Form a "rose" with a distinguished vertex v_0 and one "petal" labelled by each of the generators h_i. This is not necessarily a graph labelled over $X \cup X^{-1}$ in the sense described above, because there may be two edges with the same label starting at v_0: if there are two such edges, then form a new graph by identifying these edges (one says that the edges are "folded" together). This may result in additional vertices with two edges with the same label starting at the vertex. Continue folding edges until no further folding can occur. Clearly this process stops after a finite number of steps since we started with a finite graph. It can be proved that the folding process is confluent, and also that the resulting graph is independent of the choice of generators for H. Denote the resulting graph by $\Gamma(H)$. Clearly the labelling of this graph over $X \cup X^{-1}$ is consistent with an immersion over B_X in the sense described above (we are assuming implicitly that an edge from v to w labelled by a letter $x \in X \cup X^{-1}$ has an inverse edge from w to v labelled by x^{-1} of course). One may view the graph $\Gamma(X)$ as the "core" graph that is obtained from the coset graph of H in $FG(X)$ by pruning trees off the coset graph.

These ideas are illustrated in the following simple examples, where $\Gamma_i = \Gamma(H_i)$ for $i = 1, 2, 3$.

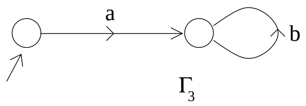
Example 1. $H_1 = \langle aba^{-1}b^{-1}, a^2, aba, ba^2 \rangle \leq FG(a, b)$.



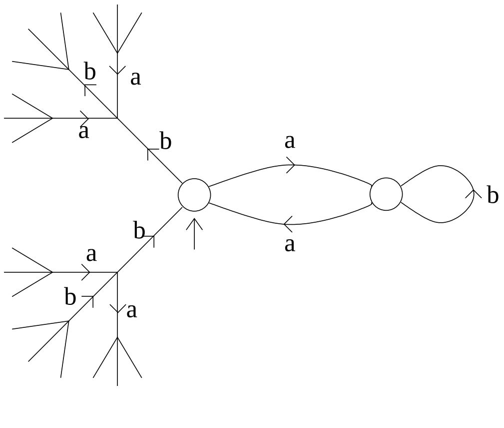
Example 2. $H_2 = \langle aba^{-1}, a^2 \rangle \leq FG(a, b)$.



Example 3. $H_3 = \langle aba^{-1} \rangle \leq FG(a, b)$.



Example 2 (continued). Note that the graph $\Gamma(H_2)$ may be obtained from the right coset graph of H_2 in $FG(a, b)$ by pruning trees off this graph as illustrated below.



One may regard $\Gamma(H)$ as an automaton with initial and terminal state equal to the image of the distinguished vertex v_0 in the folded graph. The distinguished state (vertex) of each automaton $\Gamma(H_i)$ for the subgroups H_i of examples 1, 2 and 3 are indicated by arrows at the corresponding vertices. From this point of view, we can study the *language* accepted by $\Gamma(H)$, namely $L(\Gamma(H)) = \{w \in (X \cup X^{-1})^* : w$ labels a path in $\Gamma(H)$ starting and ending at $v_0\}$. The set of reduced words in this language coincides with the set of reduced words representing elements of H, i.e. the image of this language in $FG(X)$ is just H. The image of this language in $FIM(X)$ is a closed inverse submonoid of $FIM(X)$, which we will denote by $\bar{H}$. Conversely, if Γ is any finite connected graph whose edges are labelled over $X \cup X^{-1}$ consistent with an immersion to B_X and if v_0 is a fixed base point of $V(\Gamma)$, then by viewing Γ as an automaton with initial and terminal state v_0, the language accepted by this automaton gives rise to a finitely generated subgroup H of $FG(X)$ and a finitely generated closed inverse submonoid $\bar{H}$ of $FIM(X)$. If we change the base point, the associated subgroups (closed inverse submonoids) are conjugate. Much additional information about closed inverse submonoids of free inverse monoids is provided in the thesis of Ruyle [122].

It is natural from an automaton point of view to also consider the *syntactic monoid* of this automaton. Each letter $x \in X$ induces a partial one-one map (denoted again by X) on the set $Q = V(\Gamma(H))$ of vertices of $\Gamma(H)$. The letter x maps vertex v to vertex w iff there is an edge labelled by x from v to w: clearly x^{-1} is the inverse partial one-one map: x^{-1} maps w to v iff x maps v to w. The submonoid of $SIM(Q)$ generated by these partial one-one maps induced by the letters in $X \cup X^{-1}$ is an inverse monoid, which we denote by $I(H)$ and refer to as the *syntactic monoid* of H. Clearly $I(H)$ is finite because $\Gamma(H)$ is finite.

Examples 1, 2 and 3 (continued). Consider the subgroups $H_1 = \langle aba^{-1}b^{-1}, a^2, aba, ba^2\rangle$, $H_2 = \langle aba^{-1}, a^2\rangle$, and $H_3 = \langle aba^{-1}\rangle$ of $FG(a, b)$. It is easy to check from the resulting automata $\Gamma(H_i)$ displayed above that $I(H_1) \cong \mathbf{Z}_2$ (the cyclic group of order 2), while $I(H_2) \cong SIM(a, b)$ and $I(H_3)$ is isomorphic to the combinatorial Brandt monoid of order 6 (i.e. the monoid consisting of the 2×2 matrix units together with the zero and identity matrix).

The graph $\Gamma(H)$ associated with a finitely generated subgroup of a free group $FG(X)$ has been used by many authors to study various properties of the subgroup H. For example, this construction enables one to quickly determine the rank of H and to find a free basis for H, to solve the membership problem for H in $FG(X)$, to solve the conjugacy problem for finitely generated subgroups of $FG(X)$, to give a quick proof of Marshall Hall's theorem that shows that finitely generated subgroups of free groups are closed in the profinite topology [Hall], to determine whether H has finite index in $FG(X)$, to determine whether H is normal in $FG(X)$, to determine whether H is malnormal in $FG(X)$, to give a quick proof of Howson's theorem [62], and of Takashasi's theorem [143], to study algebraic extensions of subgroups of free groups, and to study the well-known conjecture of Hanna Neumann about the rank of the intersection of two finitely generated subgroups of a free group. We refer to the paper by Kapovich and Myasnikov [68] for many references to the literature and a survey of some of the results that have been obtained for subgroups of free groups via Stallings foldings.

While many results about finitely generated subgroups of free groups may be obtained just by studying the graphs $\Gamma(H)$ directly, there are other results that require an analysis of the structure of the syntactic monoid $I(H)$. The structure of this (finite) inverse monoid provides some information about how the subgroup H sits inside the free group $FG(X)$, and may be used to analyze the complexity of some algorithmic problems about finitely generated subgroups of free groups. Recall that a subgroup H of a group G is called a *pure* subgroup if $g^n \in H$ for some integer $n > 1$ implies $g \in H$. The second statement in the following theorem is proved in the paper of Birget, Margolis, Meakin, and Weil [20]: the first statement is a slight restatement of a well known fact (see [86] or [68]).

Theorem 14 *Let $H = \langle h_1, h_2, \ldots, h_n \rangle$ be a finitely generated subgroup of a free group $FG(X)$. Then*
(1) H has finite index in $FG(X)$ iff $I(H)$ is a group (iff $\Gamma(H)$ is a cover over B_X). One can decide whether $h_1, h_2, \ldots, h_n$ generate a finite index subgroup of $FG(X)$ in polynomial time.
(2) H is a pure subgroup of $FG(X)$ iff $I(H)$ is combinatorial (i.e. all maximal subgroups of $I(H)$ are trivial). The problem of deciding whether $h_1, h_2, \ldots, h_n$ generate a pure subgroup of $FG(X)$ is PSPACE-complete.

Thus for the subgroups H_1, H_2, H_3 of $FG(a, b)$ considered in Examples 1, 2 and 3 above, we see that H_1 is of finite index in $FG(a, b)$ but is not a pure subgroup, H_2 is not of finite index and is not a pure subgroup, and H_3 is a pure subgroup but is not of finite index.

The complexity analysis involved in the proof of part (2) of the theorem above involves a series of reductions eventually to a theorem of C. Bennett [13] about the space complexity of injective Turing machines, and shows that several other natural problems about finite immersions are PSPACE complete (see [20] for details). In general terms, if an algorithmic problem about finitely generated subgroups of free groups can be decided by just examining properties of the graphs $\Gamma(H)$ directly, then these problems are usually "easy" from a complexity point of view, while they tend to be "hard" from this point of view if computation of the syntactic monoid $I(H)$ is required.

The classification of connected immersions via closed inverse submonoids of free inverse monoids and several of the associated ideas developed in this section have been extended by Delgado, Margolis and Steinberg [40] to study subgroups of arbitrary finitely presented groups: in particular, they use inverse-monoid theoretic methods to study quasiconvexity and separability properties of subgroups of groups. Similar techniques are used by Gitik [48],[49] to study separability properties and quasiconvexity in hyperbolic groups and by Steinberg [135],[136],[137] to study monoid presentations and profinite topologies on free groups. Methods related to Stallings foldings, making use of 2-complexes instead of graphs, have been employed by McCammond and Wise [97] to study coherence in groups and by Schupp [127] to study surface groups and Coxeter groups of extra large type. Kapovich, Weidmann and Myasnikov [69] have extended the methods of Stallings foldings to study subgroups of fundamental groups of graphs of groups. Recently, Luda Markus-Epstein [92] has used an extension of the methods of Stallings foldings to study algorithmic problems for finitely generated subgroups of amalgamated free products of finite groups. Her construction is related to a construction used by Cherubini, Meakin and Piochi [27] to prove that the word problem for amalgamated free products of finite inverse semigroups in the category of inverse semigroups is decidable: this is in contrast to a result of Sapir [123] that shows that the word problem for amalgamated free products of finite semigroups (in the category of semigroups) is undecidable in general. It seems plausible that further investigation of these techniques might prove fruitful in the study of algorithmic problems for subgroups of finitely presented groups and for other algorithmic problems about groups and semigroups. Indeed, a construction that employs Stallings foldings will be employed in the final section of this paper to study presentations of inverse monoids.

6 Finite Inverse Monoids and Infinite Groups

The theory of finite monoids is very closely related to the theory of finite automata and the theory of regular languages. Indeed Kleene's theorem shows that if A is a finite alphabet, then a language $L \subset A^*$ is regular (equivalently rational) if and only if its syntactic monoid is finite. Eilenberg's variety theorem [112], sets up a one-one correspondence between pseudovarieties of finite monoids and varieties of regular languages. This has led to an intensive study of pseudovarieties of finite semigroups (monoids): this is one of the central areas of research in finite semigroup theory. Inspired by the connection between finite semigroups and finite automata, Krohn and Rhodes [73] proved the "prime decomposition theorem" for finite semigroups. This can be formulated to show that every finite semigroup S is a homomorphic image of a subsemigroup of some iterated wreath product $A_1 \circ G_1 \circ A_2 \circ G_2 \ldots \circ G_n \circ A_{n+1}$, where each A_i is a semigroup with only trivial subgroups and each G_i is a group. The smallest such integer n for which S can be so expressed is called the *group complexity* of S, and the problem of algorithmically deciding the group complexity of a finite semigroup is a problem of central importance in the theory. There are several books and survey articles devoted to the theory of finite semigroups and connections between this theory and the theory of profinite groups and profinite monoids (for example, Almeida [4], Pin [112], Eilenberg [45], Henckell, Margolis, Pin and Rhodes [59], Straubing [141], Weil [145],...). In the present section, I will discuss briefly some results of Ruyle [122] linking pseudovarieties of finite inverse monoids and the theory of equations in free groups, and I will discuss (again briefly) some connections between finite inverse monoids, finite immersions, and profinite topologies on free groups.

A *pseudovariety* of finite [inverse] monoids is a class $\mathbf{V}$ of finite [inverse] monoids that is closed under taking [inverse] submonoids, homomorphic images and finite direct products. It is well-known that if K is a set of finite [inverse] monoids, then the pseudovariety of finite [inverse] monoids generated by K (i.e. the smallest pseudovariety of [inverse] monoids containing K) is $\langle K \rangle = HSP(K)$, the class of [inverse] monoids that are homomorphic images of [inverse] submonoids of finite direct products of monoids in K. A question of interest in finite semigroup theory is the question of decidability of membership in a pseudovariety $\mathbf{V}$: i.e., given a pseudovariety $\mathbf{V}$ or a set K of finite [inverse] monoids, and a finite [inverse] monoid M, is there an algorithm to decide whether or not $M \in \mathbf{V}$ or $M \in \langle K \rangle$.

In his thesis [122], Ruyle developed an Eilenberg-type variety theorem linking pseudovarieties of finite inverse monoids and "varieties" of edge-labelled graphs. A *variety* of (edge-labelled) graphs is a function ν that associates with each set X a set $\nu(X)$ of finite connected $X \cup X^{-1}$-labelled graphs such that:

(1) If $\Gamma, \Lambda \in \nu(X)$, then each connected component of $\Gamma \times \Lambda$ is in $\nu(X)$.

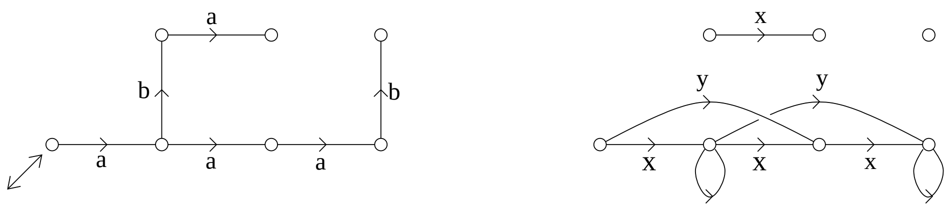
(2) If $\Gamma \in \nu(X)$ and $\Gamma \to \Lambda$ is a covering of graphs, then $\Lambda \in \nu(X)$.

(3) If $\Gamma \in \nu(Y)$ and $\psi : FIM(X) \to FIM(Y)$ is a morphism between free inverse monoids, then each component of $\Gamma\psi^{-1}$ is in $\nu(X)$.

Here the *product* $\Gamma \times \Lambda$ of two $X \cup X^{-1}$-labelled graphs is the graph with set $V(\Gamma \times \Lambda) = V(\Gamma) \times V(\Lambda)$ of vertices and with and an edge labelled by $x \in (X \cup X^{-1})$ from (u_1, u_2) to (v_1, v_2) iff there is an edge labelled by x from u_1 to v_1 in Γ and an edge labelled by x from u_2 to v_2 in Λ. Also, if $\psi : FIM(X) \to FIM(Y)$ is a morphism between free inverse monoids and if Γ is a $Y \cup Y^{-1}$-labelled graph, then $\psi^{-1}(\Gamma)$ is the $X \cup X^{-1}$-labelled graph with the same set of vertices as Γ and with an edge labelled by $x \in X \cup X^{-1}$ from v_1 to v_2 iff $\psi(x)$ labels a path from v_1 to v_2 in Γ.

For example, if $X = \{x, y, z\}$ and $Y = \{a, b\}$ and $\psi : FIM(X) \to FIM(Y)$ is defined by $x \to a, y \to a^2, z \to bb^{-1}$, then the result of applying inverse edge substitution ψ^{-1} to a graph

Γ is shown in the diagram below.



Γ $\Gamma\psi^{-1}$

For Γ a finite connected $X \cup X^{-1}$-labelled graph, denote by $I(\Gamma)$ the syntactic inverse monoid of Γ (i.e. the inverse monoid of partial one-one maps on the set of vertices of Γ induced by the letters of X). The following theorem was proved by Ruyle [122] in 1997.

Theorem 15 *(Ruyle) There is a one-one correspondence between pseudovarieties of finite inverse monoids and varieties of finite connected edge-labelled graphs, given by*
$\mathbf{V} \to \nu$ *(where* $\nu(X)$ *is the set of* $X \cup X^{-1}$*-labelled graphs* Γ *such that* $I(\Gamma) \in \mathbf{V}$*),*
and
$\nu \to \mathbf{V} = \langle I(\Gamma) : \Gamma \in \nu(X),$ *some* $X\rangle$.

Now denote by $\mathcal{T}(X)$ the class of all $X \cup X^{-1}$-labelled Munn trees, and by $\langle \mathcal{T} \rangle$ the variety of all graphs generated by all Munn trees. Denote by $\mathbf{T}$ the corresponding pseudovariety of finite inverse monoids. In his thesis [122], Ruyle showed that $\langle \mathcal{T} \rangle = CPIn(\mathcal{T})$, i.e. every graph in $\langle \mathcal{T} \rangle$ is covered by some finite product of inverse images of Munn trees. He also proved the following interesting connection between membership in $\langle \mathcal{T} \rangle$ (or equivalently in $\mathbf{T}$) and the theory of equations in free groups.

Theorem 16 *(Ruyle) The membership problem in* $\langle \mathcal{T} \rangle$ *is algorithmically equivalent to deciding consistency of coefficient-free systems of equations and inequations* $(u \neq v)$ *in free groups.*

An immediate corollary of this and Makanin's work [83] on decidability of the universal theory of free groups is the following:

Corollary 2 *There is an algorithm to decide membership in* $\langle \mathcal{T} \rangle$.

An important result in finite semigroup theory is a theorem of Ash [7] that proves that the pseudovariety of finite semigroups generated by the class of finite inverse semigroups is precisely the class of finite semigroups whose idempotents commute: thus membership in the pseudovariety of semigroups generated by the finite inverse semigroups is decidable.

Ash also provided the solution to another celebrated problem in finite semigroup theory with his solution [8] to the Rhodes "Type II" conjecture. An alternative proof of this conjecture was provided by Ribes and Zalesskiĭ [119], using methods of profinite groups. It is a well-known result of Marshall Hall [57] that a finitely generated subgroup of a free group is closed in the profinite topology: Ribes and Zalesskiĭ showed that the product $H_1 H_2 \ldots H_n$ of finitely many finitely generated subgroups of a free group is closed in the profinite topology, and this implies that Rhodes' "Type II" conjecture is true (see the survey paper by Henckell, Margolis, Pin and Rhodes [59] for connections between these problems).

Let $\mathbf{V}$ be a pseudovariety of finite groups. Recall [57] that in the pro-$\mathbf{V}$ topology on a group G, a basis of clopen neighborhoods of 1 is given by the normal subgroups N of G such that N

has finite index in G and $G/N \in \mathbf{V}$. (The profinite topology on G corresponds to the case where the pseudovariety $\mathbf{V}$ consists of all finite groups of course.) A pseudovariety $\mathbf{V}$ of groups is said to be *extension-closed* if every finite extension of a group in $\mathbf{V}$ is also in $\mathbf{V}$.

Recall that if H is a finitely generated subgroup of a free group $F(X)$, then the syntactic monoid $I(H)$ of H is the inverse monoid of partial one-one maps on the vertices of the associated immersion $\Gamma(H)$ induced by the letters in X. We say that H *is* $\mathbf{V}$*-extendible* if the immersion $\Gamma(H)$ can be embedded in a finite cover over B_Y for some finite set Y containing X in such a way that the syntactic monoid of this cover is a group in $\mathbf{V}$. That is, the set of partial one-one maps in $I(H)$ can be extended to a set of permutations on a possibly larger set of states generating a group in $\mathbf{V}$. This is equivalent to saying that $I(H)$ has an E-unitary cover over some finite group in $\mathbf{V}$. The following result was proved by Ribes and Zalesskiǐ [120] and by Margolis, Sapir and Weil [91].

Theorem 17 *Let H be a finitely generated subgroup of a free group $FG(X)$. If $\mathbf{V}$ is extension-closed, then the following are equivalent*

(1) H is closed in the pro-$\mathbf{V}$ topology.

(2) H is a free factor of a clopen subgroup of $FG(X)$.

(3) H is $\mathbf{V}$-extendible.

In [91], Margolis, Sapir and Weil provided an algorithm for computing the pro-p closure and the pro-nil closure of a finitely generated subgroup of a free group $FG(X)$. The question of providing an algorithm to compute the pro-solvable closure of a finitely generated subgroup of a free group remains open as far as I am aware. For additional information about the extension problem for finite inverse monoids of partial one-one maps, the reader is referred to the paper by Steinberg [137] and the paper by Auinger and Steinberg [9].

7 Presentations of Inverse Monoids

The inverse monoid M presented by a set X of generators and relations of the form $u_i = v_i$ will be denoted by $M = Inv\langle X : u_i = v_i\rangle$. Here $u_i, v_i \in (X \cup X^{-1})^*$. Clearly M is the image of the free inverse monoid $FIM(X)$ by the congruence generated by the defining relations, or equivalently, it is the quotient of $(X \cup X^{-1})^*$ obtained by applying the relations $u_i = v_i$ and the identities that define the variety of inverse monoids. It is easy to see that $G = Gp\langle X : u_i = v_i\rangle$ is the maximal group homomorphic image of the inverse monoid M with the same presentation as an inverse monoid. For example, the bicyclic monoid has a presentation $Inv\langle a : a^{-1}a = 1\rangle$ as an inverse monoid (and its maximal group image is $\mathbf{Z}$), a polycyclic monoid may be defined by $Inv\langle X : x^{-1}x = 1,\ x^{-1}y = 0,\ x \neq y\rangle$ (and it has trivial maximal group image), and the free group $FG(X)$ may be defined by $FG(X) = Inv\langle X : xx^{-1} = x^{-1}x = 1 \text{ for all } x \in X\rangle$. Of course $FIM(X) = Inv\langle X : \emptyset\rangle$ and its maximal group image is $FG(X)$.

If $P = Mon\langle X : u_i = v_i\rangle$ embeds in $G = Gp\langle X : u_i = v_i\rangle$, then we have associated two inverse semigroups with this situation: the inverse hull of P, and the Toeplitz inverse monoid $T(G, P)$ (see section 3 of this paper to recall the definitions). We clearly have $\beta_{x^{-1}}\beta_x = 1$ in the Toeplitz inverse monoid $T(G, P)$. This expresses the fact that each $\beta_x, x \in X$ has domain P. Clearly if $\beta_g \in T(G, P)$ has domain P then $g \in P$. This simply says that P embeds as the $\mathcal{L}$-class of 1 in $T(G, P)$. If we identify each $x \in X$ with $\beta_x \in T(G, P)$, then $u_i = v_i$ is a relation in $T(G, P)$ as well. However, even if $T(G, P)$ is bisimple (as is the case for example for the bicyclic monoid, or if G is a Garside group etc - see sections 2 and 3), then $T(G, P)$ has in general other relations that are not obviously consequences of the defining relations for P.

For example, if $P = Mon\langle a, b : ab = ba\rangle$ (the free commutative monoid on 2 generators), then P embeds as the positive cone of the free abelian group of rank 2, $PP^{-1} = G$, and principal left ideals of P form a lattice, so $T(G, P)$ is bisimple and coincides with the left inverse hull $I_l(P)$. However, a presentation for $T(G, P)$ in this case is given by $T(G, P) = Inv\langle a, b : a^{-1}a = b^{-1}b = 1,\ ab = ba,\ ab^{-1} = b^{-1}a\rangle$, i.e. $T(G, P)$ is a direct product of two bicyclic monoids. It will be apparent with the results to be described in this section that the relation $ab^{-1} = b^{-1}a$ is not a consequence of the other defining relations for this inverse monoid. More generally, one can provide a presentation for the Toeplitz inverse monoid associated with the embedding of a partially commutative monoid in its corresponding right angled Artin group. Recall that a *Coxeter matrix* is a matrix $(m_{x,y}), x, y \in X$ such that $m_{x,y} = m_{y,x}, m_{x,x} = 1$ and $m_{x,y} \in \{2, \infty\}$. The associated *right-angled Artin group* is the group $G = Gp\langle X : xy = yx \text{ if } m_{x,y} = 2\rangle$ and the corresponding partially commutative monoid P with the same presentation as a monoid embeds in G. This is well-known, and follows from the theorem on Paris (Theorem 4 above) of course). The corresponding Toeplitz inverse monoid $T(G, P)$ has presentation $T(G, P) = Inv\langle X : x^{-1}x = 1,\ xy = yx,\ x^{-1}y = yx^{-1} \text{ if } m_{x,y} = 2,\ x^{-1}y = 0 \text{ if } m_{x,y} = \infty,\ x, y \in X\rangle$. This presentation is implicit in the paper of Crisp and Laca [33]. Note that $PP^{-1} \neq G$ in general for right-angled Artin groups, in fact from Nica's theorem (Theorem 8 above), a right-angled Artin group G is the group of (right) quotients of its associated partially commutative monoid P if and only if it is a free abelian group (i.e. $m_{x,y} = 2$ for all $x \neq y$).

It is of interest to study presentations for Toeplitz inverse monoids in general. If $PP^{-1} \neq G$ then by Theorem 8, $T(G, P)$ has a zero and there will be relations similar to the relations in the polycyclic monoid above. If $PP^{-1} = G$, then $T(G, P)$ is F-inverse with maximal group image G, so the relations in $T(G, P)$ must be consequences of the defining relations in the group G.

There are several other natural ways to associate inverse monoids with monoid presentations $P = Mon\langle X : u_i = v_i\rangle$ that embed in the associated group with the same presentation. For example, the inverse monoids $S = Inv\langle X : u_i = v_i\rangle$ and $T = Inv\langle X : u_i^{-1}v_i = 1\rangle$ (or its dual) and $M = Inv\langle X : x^{-1}x = 1, u_i^{-1}v_i = 1\ \forall x \in X\rangle$ (or its dual) are clearly all inverse monoids with maximal group image G. Clearly there are surjective morphisms $S \to T \to M$. As will be apparent from some of the results discussed later in this section, these inverse monoids are not isomorphic in general. In good cases, one or more of these monoids may be E-unitary, and it may be possible to exploit this fact to study the group G, although little work has been done along these lines as far as I am aware.

Many other interesting examples of presentations of inverse monoids arise in the theory of C^*-algebras. For example, Hancock and Raeburn [58] provide a presentation for the *Cuntz-Krieger* inverse monoid associated with the well-known Cuntz-Krieger algebra $\mathcal{O}_A$ associated with an $n \times n$ zero-one matrix $A = (A_{i,j})$ with at least one non-zero entry in each row and column. The corresponding Cuntz-Krieger inverse monoid has presentation $C_A = Inv\langle s_1, \ldots, s_n : s_i^{-1}s_j = 0,\ i \neq j,\ s_i^{-1}s_is_js_j^{-1} = A_{i,j}s_js_j^{-1}\ \forall i, j\rangle$. We refer to [58] for details and to the book of Paterson [110] or the extensive literature on graph C^*-algebras for indications of many more examples of inverse monoids arising in connection with C^*-algebras. Other interesting examples of inverse semigroups arise in connection with the study of tilings (see for example the work of Kellendonk and Lawson [71]) and in the study of self-similar actions of inverse monoids (see the paper by Bartholdi, Grigorchuk, and Nekrasevych [11] for an introduction to this subject). It would be of interest to study such semigroups from the point of view of presentations.

The main tool that has been exploited to study the word problem for presentations of inverse monoids is the construction of the associated Schützenberger graphs, initiated by Stephen [139]. If $M = Inv\langle X : u_i = v_i\rangle$, then one may consider the corresponding Cayley graph $\Gamma(M, X)$. This graph has the elements of the inverse monoid M as vertices and has an edge labelled by

$x \in X \cup X^{-1}$ from m to mx for each $m \in M$. One may view this edge as starting at m, ending at mx, and labelled by x. However, the Cayley graph $\Gamma(M, X)$ suffers from the deficiency that it is not strongly connected in general (unless M happens to be a group) - there is not necessarily an edge labelled by x^{-1} starting at mx and ending at m: as an extreme example, if the monoid M happens to have a zero, then there may be many edges in this graph ending at zero, but all edges that start at zero must obviously end at zero. Thus the Cayley graph is not an X-labelled graph in the sense defined in section 5 of this paper. Unlike the situation for Cayley graphs of groups, the Cayley graph of an inverse monoid is not naturally equipped with a word metric that turns it into a geodesic metric space.

To overcome this difficulty, it is convenient to study the *strongly connected components* of $\Gamma(M, X)$. A strongly connected component of the Cayley graph $\Gamma(M, X)$ corresponds exactly to the restriction of $\Gamma(M, X)$ to an $\mathcal{R}$-class of M. Such graphs are referred to in the literature at the *Schützenberger graphs* associated with M by virtue of their connection with the Schützenberger representation of M. In more detail, for each word $u \in (X \cup X^{-1})^*$, we denote by $S\Gamma(M, X, u)$ the graph with set $R_u = \{m \in M : mm^{-1} = uu^{-1} \text{ in } M\}$ of vertices, and with an edge labelled by $x \in X \cup X^{-1}$ from m to mx if $m, mx \in R_u$. It is not difficult to see that if x labels an edge from m to mx in $S\Gamma(M, X, u)$, then x^{-1} labels an edge from mx to m in this graph. Thus the Schützenberger graphs $S\Gamma(M, X, u)$ are strongly connected subgraphs of the Cayley graph $\Gamma(M, x)$: in fact it is not difficult to see that these graphs are geodesic metric spaces with respect to the word metric, similar to the situation for Cayley graphs of groups.

There is a Schützenberger graph corresponding to each idempotent of the inverse monoid M and there is an obvious natural graph morphism (not necessarily an injection) from each Schützenberger graph to the Cayley graph $\Gamma(G, X)$ where $G = Gp\langle X : u_i = v_i\rangle$ is the maximal group image of M. It is not very difficult to show that the inverse monoid M is E-unitary if and only if each Schützenberger graph of M naturally *embeds* in the Cayley graph $\Gamma(G, X)$. This provides an appropriate geometric interpretation of the notion of an E-unitary inverse monoid that has will be exploited later in this section.

It is useful to consider the *Schützenberger automaton* $\mathcal{A}(M, X, u) = (uu^{-1}, S\Gamma(M, X, u), u)$ with initial state (vertex) $uu^{-1} \in M$, terminal state $u \in M$ and set $S\Gamma(M, X, u)$ of states. From this point of view, the *language* accepted by this automaton is the set

$$L(u) = \{v \in M : v \text{ labels a path in } \mathcal{A}(M, X, u) \text{ from } uu^{-1} \text{ to } u \text{ in } S\Gamma(M, X, u)\}.$$

(Here we are interpreting u and v both as words in $(X \cup X^{-1})^*$ and as elements of M, and we are regarding the language $L(u)$ as a subset of M. Strictly speaking, if $v \in (X \cup X^{-1})^*$, then the corresponding element of M is $\tau(v)$ where τ is the natural homomorphism from $(X \cup X^{-1})^*$ to M, but I will suppress this notation for ease of exposition: the context makes it clear whether v is being considered as a word in $(X \cup X^{-1})^*$ or as an element of M.)

The following result of Stephen [139] is central to the theory.

Theorem 18 *(Stephen)*

Let $M = Inv\langle X : u_i = v_i\rangle$ and let $u, v \in (X \cup X^{-1})^*$ (also interpreted as elements of M as above). Then

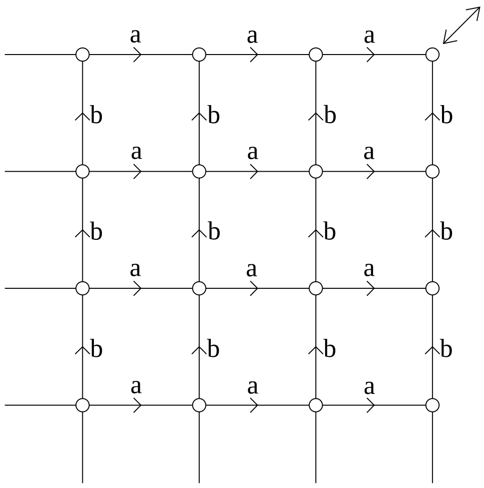
(1) $L(u) = \{v \in M : v \geq u$ in the natural partial order on $M\}$.

(2) $u = v$ in M iff $L(u) = L(v)$ iff $u \in L(v)$ and $v \in L(u)$ iff $\mathcal{A}(u)$ and $\mathcal{A}(v)$ are isomorphic as birooted edge-labelled graphs.

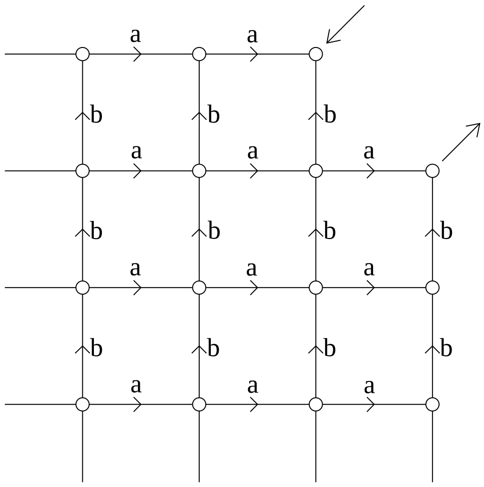
Thus we are able to solve the word problem for M if we can effectively construct the automata $\mathcal{A}(M, X, u)$ for each $u \in (X \cup X^{-1})^*$. This is of course analogous to solving the word problem for a group presentation by effectively constructing the corresponding Cayley graph. In his paper [139], Stephen provided an iterative procedure for constructing the Schützenberger automaton of a word $u \in (X \cup X^{-1})^*$, analogous to the Todd-Coxeter procedure for iteratively constructing the Cayley graph of a group presentation: I will briefly review his construction. Start with the "linear automaton" of the word u, i.e. the $X \cup X^{-1}$-labelled graph with the word u labelling a linear path from an initial vertex α to a terminal vertex ω. Build intermediate automata by successively applying one of the following two operations: (1) Stallings foldings, and (2) "expansions". Here an expansion is described as follows. If in an intermediate automaton there is a path from some vertex γ to some vertex δ labelled by one side (say u_i) of one of the defining relations in M, then sew on (add) a new path from γ to δ labelled by the other side (v_i) of the defining relation in M. Stephen shows in [139] that these operations are confluent, that there is a well-defined process of closing with respect to these two operations, and that the resulting process limits (in a sense made precise in his paper) to the Schützenberger automaton $\mathcal{A}(M, X, u)$.

For example, if $M = Inv\langle X : \emptyset\rangle$, the free inverse monoid on X, then no expansions apply: it is easy to see that if one starts with the linear automaton of a word u, then successive applications of Stallings foldings produces the Munn tree $MT(u)$ (which coincides with the Schützenberger automaton in this case), so Munn's theorem (Theorem 9) follows as a special case of Stephen's results, as would be expected.

As another example, consider the inverse monoid presentation $M_1 = Inv\langle a, b : a^{-1}a = b^{-1}b = 1, ab = ba\rangle$. If we start with a single vertex (call it 1), then at this vertex it is possible to read the empty path (i.e. the path labelled by 1), so an expansion can be applied and we sew on a path labelled by $a^{-1}a$ starting and ending at the vertex 1. An application of a Stallings folding collapses this to a single edge labelled by the letter a ending at the vertex 1 and starting at some other vertex (let's call it a^{-1}): similarly we attach an edge labelled by the letter b ending at the vertex 1 and starting at some other vertex (let's call it b^{-1}). Repeat the process at the vertex b^{-1}: we sew on an edge labelled by a^{-1} ending at the vertex b^{-1}. Let's call the initial vertex of this edge $a^{-1}b^{-1}$. This creates a path labelled by ab from $a^{-1}b^{-1}$ to 1, so we must sew in a path labelled by ba from the vertex $a^{-1}b^{-1}$ to the vertex 1. But then there are two edges labelled by the letter a ending at the vertex 1, so we identify these by a Stallings folding, to obtain a square as indicated in the diagram below. Repetition of this argument at any vertex shows that the Schützenberger graph of 1 consists of the third quadrant of the $a - b$ plane as shown in the diagram below.



Similarly, if u is any word, the Schützenberger graph of u is closed under adding all squares of the Cayley complex of the free abelian group on $\{a, b\}$ to the "south-west" of any vertex constructed. One sees that all Schützenberger graphs of this monoid embed in the Cayley graph of the free abelian group on $\{a, b\}$, so M_1 is E-unitary. For example, the Schützenberger graph of the word $b^{-1}a$ is depicted in the diagram below.



Note that $ab^{-1} \notin L(b^{-1}a)$ so $ab^{-1} \neq b^{-1}a$ in the inverse monoid M_1. On the other hand, if one considers the Toeplitz inverse monoid $M_2 = Inv\langle a, b : a^{-1}a = b^{-1}b = 1,\ ab = ba,\ a^{-1}b = ba^{-1}\rangle$, then the Schützenberger graphs of M_2 are subgraphs of the Cayley graph of the free abelian group on $\{a, b\}$ that are closed under addition of squares to the "south-west" of any vertex as above, but also are closed under completing all squares of the Cayley complex whenever two consecutive edges of a square are present. Thus there is no word whose Schützenberger graph in M_2 is equal to the Schützenberger graph of $b^{-1}a$ in M_1. It follows that the inverse monoids M_1 and M_2 are not isomorphic.

Schützenberger automata have played a pivitol role in much of the theory of presentations of inverse monoids. In his paper [139], Stephen used these automata to solve the word problem for the free inverse monoid on n commuting generators (this monoid is not E-unitary if $n > 2$, but it is E-unitary if $n = 2$ [95]). Margolis and Meakin [87] made use of these automata, together with Rabin's tree theorem, to solve the word problem for inverse monoids of the form $Inv\langle X : e_i = f_i,\ i = i, \ldots, n\rangle$ where e_i and f_i are Dyck words (i.e. idempotents of $FIM(X)$). The automata were used by several authors to study free products of inverse semigroups [67], and various classes of amalgamated free products and HNN extensions in the category of inverse semigroups (e.g [14], [65], [26], [27]..) Related work on the structure of amalgams and HNN extensions of inverse semigroups has been done by Haataja, Margolis and Meakin [56], Bennett [15], Yamamura [146], [147], and Gilbert [47].

The notion of the *Schützenberger complex* of a presentation of an inverse monoid has been introduced by Steinberg [138] by analogy with the corresponding notion of the Cayley complex of a group presentation: this notion has been used by Lindblad [78] in his work on the prefix membership problem for one-relator groups. Margolis, Meakin and Stephen [89] used Schützenberger automata to solve the word problem for finitely generated free Burnside inverse semigroups in the variety of inverse semigroups defined by an identity of the form $w^a = w^{a+b}$ where $b \leq a$: the word problem for free Burnside inverse semigroups in the corresponding variety where $b > a$ remains unknown as far as I am aware. [As an aside, it is interesting to note that the word problem for finitely generated free Burnside semigroups in the variety of *semigroups* defined by the identities $w^a = w^{a+b}$ is known for all values of $a, b > 0$ except $a = 2$ and $b = 1$ - see the papers of McCammond [96], Guba [55], de Luca and Varricchio [42], and do Lago [43] for much information about Burnside varieties of semigroups.]

In [64], Ivanov, Margolis and Meakin made heavy use of Schützenberger graphs as well as van Kampen diagram arguments to study one relator inverse monoids. In view of the difficulty of solving the word problem for one-relation semigroups of the form $Sgp\langle X : u = v\rangle$ where $u, v \in X^+$, I will restrict attention to one-relator inverse monoids of the form $Inv\langle X : w = 1\rangle$. If w is a Dyck word (i.e. an idempotent in $FIM(X)$), then there is a fast algorithm provided in the paper of Birget, Margolis and Meakin [19] to solve the word problem. However, in general this problem remains unsolved. In fact, the following result in [64] shows that this problem is at least as difficult as the one-relation semigroup problem, even if we restrict to the case where w is a reduced word.

Theorem 19 *(Ivanov, Margolis, Meakin) If the word problem is decidable for every one-relator inverse monoid of the form $Inv\langle X : w = 1\rangle$, for w a reduced word, then the word problem for every one relation semigroup $Sgp\langle X : u = v\rangle$ (for $u, v \in X^+$) is decidable.*

Outline of Proof. A result of Adian and Oganessian [3] reduces the word problem for one relation semigroups to the study of the word problem for semigroups with a presentation of the form $S = Sgp\langle X : aub = avc\rangle$ where $a, b, c \in X, b \neq c$ and $u, v \in X^*$. Consider the associated one relator inverse monoid $M = Inv\langle X : aubc^{-1}v^{-1}a^{-1} = 1\rangle$. Note that $aubc^{-1}v^{-1}a^{-1}$ is a reduced word (but not a cyclically reduced word) since $b \neq c$. In [64], Ivanov, Margolis and Meakin show, by using some results of Adian and Oganessian [3] and by using the inverse hull of the semigroup S, that S embeds in M. It follows that if the word problem is decidable for M, then it must also be decidable for S. ■

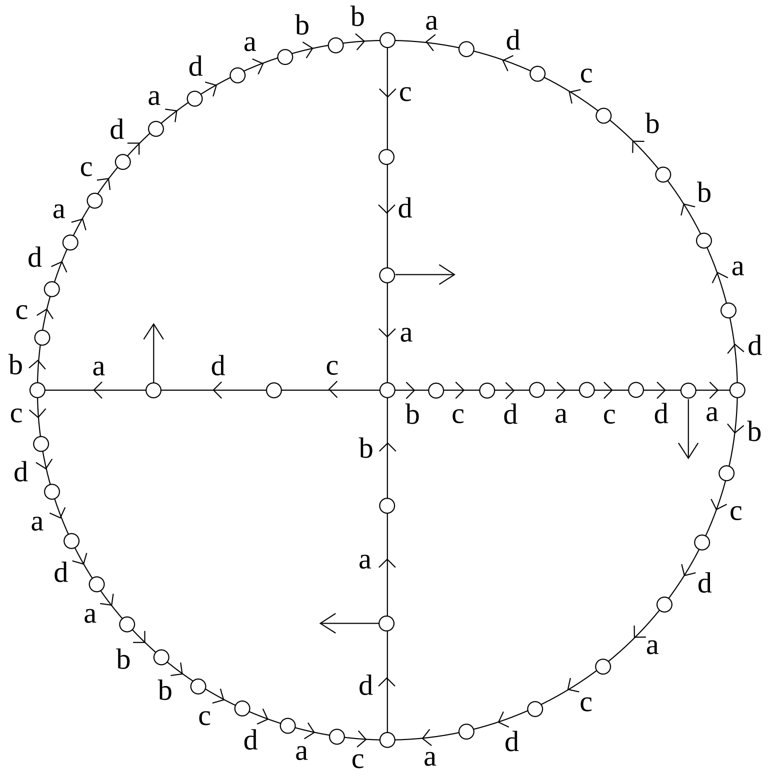
In view of this result, I will restrict attention further to the consideration of inverse monoids of the form $Inv\langle X : w = 1\rangle$ where w is a *cyclically reduced* word. In this case, Ivanov, Margolis and Meakin [64] proved the following structural result about such monoids.

Theorem 20 *(Ivanov, Margolis, Meakin) Every inverse monoid of the form $Inv\langle X : w = 1\rangle$, where w is a cyclically reduced word, is E-unitary.*

I will outline some of the arguments used in the proof of this theorem, since they are of independent interest and provide a nice blend of techniques from geometric group theory and inverse semigroup theory. First note that if w can be factored in the free monoid $(X \cup X^{-1})^*$ as $w \equiv uv$, then it does not necessarily follow that the cyclic conjugate vu of w is also 1 in the inverse monoid $M = Inv\langle X : w = 1\rangle$. (For example, we know from looking at the bicyclic monoid that $ab = 1$ does not imply that $ba = 1$). The cyclic conjugate vu of the word $w \equiv uv$ is called a *unit cyclic conjugate* of w if $vu = 1$ in M.

In order to prove that the inverse monoid M is E-unitary, it is necessary to prove that every word that is 1 in the corresponding maximal group homomorphic image $G = Gp\langle X : w = 1\rangle$ must be an idempotent of M. Now words that are 1 in G are precisely the words that label the boundary of some van Kampen diagram over G, by van Kampen's lemma (see Lyndon and Schupp [82] or Ol'shanskii [107]). Consequently, we need to show that every word labelling the boundary of every van Kampen diagram over G must be an idempotent in M. A lemma in [64] shows that this happens if, for every van Kampen diagram Δ over G, there is some cell Π of Δ and some vertex $\alpha \in \partial\Delta \cap \partial\Pi$ such that the cyclic conjugate of $w^{\pm 1}$ obtained by reading around the boundary of Π, starting at α, is a *unit* cyclic conjugate of $w^{\pm 1}$. In this situation, we say that *a unit cyclic conjugate of w starts on the boundary of* Δ. Thus the problem reduces to showing that there is a unit cyclic conjugate of w starting on the boundary of every van Kampen diagram over G. Unfortunately, we do not know of any general procedure for testing whether a cyclic conjugate of a cyclically reduced word w is a unit cyclic conjugate. In fact, the situation can be quite complicated, as the following example in a paper by Margolis, Meakin, and Stephen [88] shows.

Example 4 Let $w = abcdacdadabbcdacd$, a 17-letter word in $\{a, b, c, d\}^*$. In [88] it is shown that there is a van Kampen diagram Δ with four cells Π_i, $i = 1, \ldots, 4$ over w such that for every cell Π_i of Δ, the vertex on $\partial(\Pi_i)$ at which one reads $w^{\pm 1}$ around $\partial(\Pi_i)$ is an interior vertex of the diagram Δ. The situation is represented in the picture below, where the arrows depict vertices at which the word w is read around the corresponding cell.



Note that one may construct this diagram Δ by starting with any of the vertices labelled by arrows and sewing on a copy of a cell whose boundary is labelled by w at that vertex, then successively sewing on three more cells with boundaries labelled by w at appropriate vertices, and applying Stallings foldings. Thus the 1-skeleton of this diagram is an intermediate graph along the way to constructing the Schützenberger graph of 1 for the corresponding inverse monoid $M = Inv\langle a, b, c, d : w = 1\rangle$. It follows by Stephen's theorem (Theorem 18 above), that any word that labels a loop based at any of the interior vertices labelled by arrows must be 1 in M. In particular, the cyclic conjugates of w obtained by starting at a vertex marked by an arrow and reading around the boundary of the adjacent cell, must be a unit cyclic conjugate of w. This provides us with a method of constructing new unit cyclic conjugates of w. In particular, an analysis of this example shows that every cyclic conjugate of this word w that starts with the letter a must be a unit cyclic conjugate of w (see [88] for details). Now the Freiheitssatz tells us that the letter a must occur on the boundary of *every* van Kampen diagram over w, so there is a unit cyclic conjugate of w starting on the boundary of every van Kampen diagram over w, and this particular inverse monoid M must be E-unitary. ■

The argument outlined in this example is indicative of some of the the general arguments used in [64] to prove Theorem 20. Essential use is also made of a theorem of Ivanov and Meakin [63] about diagrammatic asphericity of a class of two-relator groups. The reader is encouraged to refer to the paper [64] for full details.

Theorem 20 implies that the Schützenberger graph $S\Gamma(M, X, 1)$ of 1 for a one-relator inverse monoid $M = Inv\langle X : w = 1\rangle$ with w cyclically reduced must embed as a subgraph of the Cayley graph $\Gamma(G, X)$ of the corresponding one-relator group G. From results of Stephen [Ste2] it follows that $S\Gamma(M, X, 1)$ is a *full* subgraph of $\Gamma(G, X)$, and also that membership in $L(u)$ is decidable for each $u \in (X \cup X^{-1})^*$ iff membership in $L(1)$ is decidable. It is clear that the set of vertices of $S\Gamma(M, X, 1)$ is precisely the set of elements in the submonoid P_w of G generated by the set of prefixes of the word w. Hence we have the following corollary to Theorem 20.

Corollary 3 *(Ivanov, Margolis, Meakin) The word problem for the inverse monoid $M = Inv\langle X : w = 1\rangle$ (for w a cyclically reduced word) is decidable if the membership problem for the submonoid P_w of the one-relator group $G = Gp\langle X : w = 1\rangle$ is decidable, i.e. iff there is an algorithm that will decide, on input an element of G, whether or not this element belongs to P_w.*

We refer to this membership problem as the *prefix membership problem* for w. Note that the prefix membership problem depends on the specific word w: a cyclic conjugate of w gives rise to the same group G but a different inverse monoid M and a different submonoid P_w of G. In general, the prefix membership problem for cyclically reduced words remains open, but several special cases of the problem have been solved. The following theorem collects some of the results about this problem that are proved in the papers by Ivanov, Margolis and Meakin [64], and the paper by Margolis, Meakin and Šuniḱ [90].

Theorem 21 *Let $G = Gp\langle X : w = 1\rangle$ where w is a cyclically reduced word and let P_w be the prefix monoid of w in G. Then the membership problem for P_w is decidable in the following cases:*

(1) Some letter of $a^{\pm 1}$ occurs only once in w ([64]);

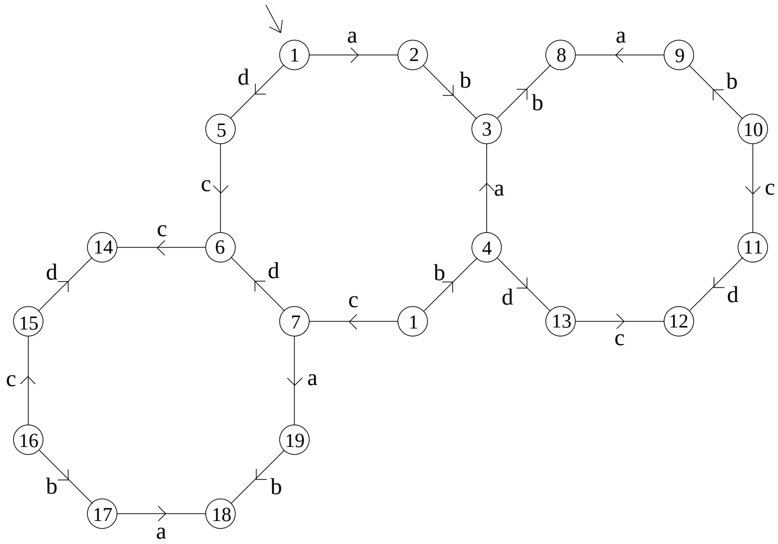
(2) w is a cyclic permutation of $[a_1, b_1][a_2, b_2] \ldots [a_n, b_n]$ ([64] and also [90]);

(3) there is a morphism f from G onto $\mathbf{Z}$ such that $f(v) > 0$ for every proper prefix v of w ([64]);

(4) $w \equiv a^{n_0} b^{m_1} \ldots a^{n_{k-1}} b^{m_k} a^{n_k}$ where $exp_a(w) = 0$ and the polynomial $m_1 x^{n_0} + m_2 x^{n_0+n_1} + \ldots + m_k x^{n_0+n_1+\ldots n_{k-1}}$ has a positive real root ([90]). (This includes Baumslag-Solitar groups $w = ab^m a^{-1} b^k$ and most Adian groups $w = uv^{-1}$ where $u, v \in X^+$ and uv^{-1} is cyclically reduced, for example.)

Example 5 As an example of what is involved in solving this problem, consider the cyclically reduced word $w = aba^{-1}b^{-1}cdc^{-1}d^{-1}$ (so the corresponding one-relator group $G = \langle a, b, c, d : w = 1\rangle$ is the fundamental group of the orientable surface of genus 2). I am aware of at least four distinct ways of solving the prefix membership problem for this word w. I will outline the ideas involved in one of the solutions that I find most geometrically appealing. Consider the iterative process of constructing the Schützenberger complex of 1 inside the Cayley complex of the surface group G. Starting at the vertex 1, we sew on a 2-cell based at 1 whose boundary is labelled by the word w, then sew on 2-cells based at all 0-cells (vertices) of this 2-cell, inside the Cayley complex of G, and continue this process, iteratively constructing 2-cells based at all resulting vertices of the Cayley graph of G that are reached by this process. We are interested in knowing which vertices of this Cayley graph are eventually reached via this iterative process. The diagram below shows a portion of the Schützenberger complex of 1 corresponding to sewing on 2-cells at the vertices 1, ab, and dcd^{-1} of the Cayley graph of G.

One can show that there are 19 different cone types of vertices in this Schützenberger complex: representatives of the 19 different cone types of these vertices are marked on the diagram below: the vertex corresponding to the element $aba^{-1}b^{-1}$ of G has the same cone type as the vertex 1: all other vertices of the Schützenberger graph of 1 have the same cone type as one of these 19 vertices. This was first proved by Haataja (unpublished, circa 1990), who essentially constructed an automatic structure on this Schützenberger graph in order to solve the prefix membership problem for this word.



A detailed discussion of this example is contained in the thesis of Lindblad [78], who shows that the dual graph of this Schützenberger complex of 1 has a tree-like structure, and extends these ideas to solve the prefix membership problem for a class of cyclically reduced words that have essentially no overlaps between pieces. ■

Example 6. (Haataja) This example, recently constructed by Haataja, shows that while the Schützenberger graph of 1 for a one-relator inverse monoid $M = Inv\langle X : w = 1\rangle$ corresponding to a cyclically reduced word w embeds in the Cayley graph of the corresponding one-relator group, this embedding is not in general an isometry. Consider the word $w = abcabaccaacdda^{-1}c^{-1}$ and the corresponding inverse monoid $M = Inv\langle a, b, c : w = 1\rangle$ and group $G = Gp\langle a, b, c : w = 1\rangle$. By some results contained in the paper of Ivanov, Margolis, and Meakin [64], the monoid M has trivial group of units. Also, the group G is a small cancellation group (in fact a $1/6th$ group), so G is word hyperbolic. Now the word $u = abaccaac$ labels a path in $S\Gamma(M, \{a, b, c\}, 1)$ but its inverse, $c^{-1}b^{-1}a^{-1}cad^{-1}d^{-1}$ in G is shorter than u and does not label a path in $S\Gamma(M, \{a, b, c\}, 1)$. Thus the embedding of $S\Gamma(M, \{a, b, c\}, 1)$ in the Cayley graph $\Gamma(G, \{a, b, c\})$ is not an isometry.

It would be of interest to determine conditions under which the embedding of the Schützenberger graph of 1 for a one-relator inverse monoid embeds isometrically into the Cayley graph of the corresponding group. This does happen for the word $w = aba^{-1}b^{-1}cdc^{-1}d^{-1}$ of example 5, for example. ■

The prefix membership problem is a special case of the more general problem of deciding membership in finitely generated submonoids of a group. There seems to be little known about this problem in general, even for groups that are known to have decidable generalized word problem. It is well known that membership in rational subsets of free groups is decidable - the original proof is due to Benois [16], and alternative proofs have been provided by other authors. Thus the membership problem for finitely generated submonoids of free groups is decidable: an alternate proof of this fact was provided in the paper by Ivanov, Margolis and Meakin [64]. Membership in rational subsets of finitely generated free abelian groups is decidable, and hence membership in finitely generated submonoids of such groups is decidable (see Grunschlag [51]). Grunschlag showed that if membership in rational subsets of some class of groups is decidable, then membership in rational subsets of groups that are virtually in that class is also decidable, so membership in finitely generated submonoids of virtually free groups or virtually abelian groups is decidable. Recently, James [66] has shown that if membership in rational subsets of two groups G_1 and G_2 is decidable, then membership in finitely generated submonoids of the free product $G_1 * G_2$ is decidable: his proof is an adaptation of the proof of Mihailova [99] that the generalized word problem for $G_1 * G_2$ is decidable if the generalized word problem for each G_i is decidable. It would be of interest to find other classes of groups for which the membership problem for finitely generated submonoids is decidable.

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References

[1] S.I Adian, Defining relations and algorithmic problems for groups and semigroups, *Trudy Math. Inst. Steklov* 85 (1966).

[2] S.I. Adian, On the embeddability of semigroups in groups, English translation: Soviet Mathematics **1** (1960), Amer. Math. Soc.

[3] S.I Adian, J. Oganessian, Problems of equality and divisibility in semigroups with a single defining relation, *Mathem. Zametki* 41 (1987) 412-421.

[4] J. Almeida, "Finite Semigroups and Universal Algebra", World Scientific (1994).

[5] K. I. Appel, One-variable equations in free groups, *Proc. Amer. Math. Soc.* 19 (1968), 912–918.

[6] A.W. Anissimov and F.D. Seifert, Zur algebraischen Charakteristik der durch kontext-freie Sprachen definierten Gruppen, *Elektron. Inform. Verarb. u. Kybern* **11** (1975) 695-702.

[7] C.J. Ash, Finite semigroups with commuting idempotents, *J. Australian Math. Soc. (Series A)* **43** (1987) 81-90.

[8] C.J. Ash, Inevitable graphs: a proof of the type II conjecture and some related decision procedures, *Inter. J. Algebra and Computation* **1** (1991) 411-436.

[9] K.Auinger and B. Steinberg, On the extension problem for partial permutations, *Proc. Amer. Math. Soc.* **131** (2003), No. 9, 2693-2703 (electronic).

[10] B.A. Barnes, Representation of the l_1 -algebra of an inverse semigroup, *Trans. Amer. Math. Soc.* **218** (1976) 361-396.

[11] L. Bartholdi, R. Grigorchuk and V. Nekrashevych, From fractal groups to fractal sets, "Fractals in Graz" (2001) 25-118, Trends Math. Birkhäuser, Basel (2003).

[12] J. Barwise (Ed.) *Handbook of Mathematical Logic*, North Holland (1978).

[13] C. Bennett, Time/space tradeoffs for reversible computation, *SIAM J. of Computing* **18** (1989) 766-776.

[14] P. Bennett, Amalgamated free products of inverse semigroups, *J. Algebra* **198** (1997) 499-537.

[15] P. Bennett, On the structure of inverse semigroup amalgams, *Inter. J. Algebra and Computation* Vol 7, No. 5 (1997) 577-604.

[16] M. Benois, Parties rationnelles du groupe libre, *C. R. Acad. Sci. Paris S'er. A-B* 269:A1188-A1190 (1969).

[17] J. Berstel, "Transductions and Context-free Languages", Teubner Studienbücher (1979).

[18] J.C. Birget, The groups of Richard Thompson and complexity, *Int. J. Algebra and Computation*, Vol. 14, Nos 5 and 6 (2004) 569-626.

[19] J.-C. Birget, S. Margolis and J. Meakin, The word problem for inverse monoids presented by one idempotent relator, *Theoretical Computer Science* **123** (1994) 273-289.

[20] J.-C. Birget, S. Margolis, J. Meakin, and P. Weil, PSPACE-complete problems for subgroups of free groups and inverse finite automata, *Theoretical Computer Science* **242** (2000) No. 1-2, 247-281.

[21] J.S. Birman, "Braids, Links, and Mapping Class Groups", Annals of Math. Studies **82** Princeton Univ. Press. (1975).

[22] J.S. Birman, New points of view in knot theory, *Bull. Amer. Math. Soc. (N.S.)* **28** (1993) 253-287.

[23] E. Brieskorn and K. Saito, Artin-Gruppen und Coxeter-Gruppen, *Invent. Math.* **17** (1972) 245-271.

[24] J.W. Cannon, W.J. Floyd and W.R Parry, Introductory notes on Richard Thompson's groups, *L'Enseignement Mathématique*, **42** (1996) 215-256.

[25] R. Charney, Injectivity of the positive monoid for some infinite type Artin groups, in "Geometric Group Theory Down Under, Proceedings of a special year in geometric group theory, Canberra, Australia, 1966", ed J. Cossey, et al, W. de Gruyter, Berlin, (1999).

[26] A. Cherubini, J. Meakin and B. Piochi, Amalgams of free inverse semigroups, *Semigroup Forum 54* (1997) 199-220.

[27] A. Cherubini, J. Meakin and B. Piochi, Amalgams of finite inverse semigroups, *J. Algebra* 285 (2005) 706-725.

[28] J.R. Cho and S.J. Pride, Embedding semigroups into groups, and the asphericity of semigroups, *Inter. J. Algebra and Computation*, Vol. 3, No. 1 (1993), 1-13.

[29] A.H. Clifford, A class of d-simple semigroups, *Amer. J. Math.* 75 (1953) 547-556.

[30] A.H. Clifford and G.B. Preston, The Algebraic theory of Semigroups, *Amer. Math Soc., Math. Surveys*, Number 7, Vol 1 (1961), Vol 2 (1967).

[31] D.E. Cohen, "Combinatorial Group Theory: A Topological Approach", Cambridge University Press (1989).

[32] L. P. Comerford and C. C. Edmunds, Products of Commutators and Products of Squares in a Free Group, *Int. J. Alg. Comp.* 4.3 (1994) 469-480.

[33] J. Crisp and M. Laca, On the Toeplitz algebras of right-angled and finite-type Artin groups, J. Australian Math. Soc. **72** (2002) No. 2, 223-245.

[34] P. Dehornoy, Thin groups of fractions, "Combinatorial and geometric group theory (New York, 2000/Hoboken, NJ 2001), 95-128, *Contemp. Math.*, **296**, Amer. Math. Soc. (2002).

[35] P. Dehornoy, "Braids and Self-distributivity", Birkhäuser, Progress in Mathematics Vol 192 (2000).

[36] P. Dehornoy, Braid groups and left distributive operations, *Trans. Amer. Math. Soc.* **345-1** (1994) 115-151.

[37] P. Dehornoy and L. Paris, Gaussian groups and Garside groups: two generalizations of Artin groups, *Proc. London Math. Soc.* (3) **79** (1999) 569-604.

[38] T. Deis, "Equations in Free Inverse Monoids", Ph.D. Thesis, Univ. of Nebraska (1999).

[39] T. Deis, J. Meakin and G. Sénizergues, Equations in free inverse monoids, Preprint.

[40] M. Delgado, S. Margolis and B. Steinberg, Combinatorial group theory, inverse monoids, automata, and global semigroup theory, *Inter. J. Algebra and Computation*, Vol 12, Nos. 1 & 2 (2002) 179-211.

[41] P. Deligne, Les immeubles des groupes de tresses généralisés, *Invent. Math* **17** (1972) 273-302.

[42] A. de Luca and S. Varricchio, On non-counting regular classes, *Theoretical Computer Science* **100** (1992) 67-104.

[43] A.P. do Lago, On the Burnside semigroups $x^m = x^{n+m}$, *Inter. J. Algebra and Computation* **6** (2) (1996) 179-227.

[44] J. Duncan and A.L.T. Paterson, C^*-algebras of inverse semigroups, *Proc. Edinburgh Math. Soc.* **28** (1985) 41-58.

[45] S. Eilenberg, "Automata, Languages and Machines", Vol. B, Academic Press, New York (1976).

[46] F.A. Garside, The braid group and other groups, *Quart. J. Math. Oxford Ser. (2)* 20 (1969) 235-254.

[47] N.D. Gilbert, HNN-extensions of inverse semigrops and groupoids, *J. Algebra* **272** (2004), No.1. 27-45.

[48] R. Gitik, Graphs and separability properties of groups, *J. Algeba* **188** (1997) 125-143.

[49] R. Gitik, On quasiconvex subgroups of negatively curved groups, *J. Pure Appl. Algebra* **119** (1998) 155-169.

[50] E. Godelle and L. Paris, On singular Artin monoids, *Contemporary Math.* **372** (2005) Amer. Math. Soc., 43-57.

[51] Z. Grunschlag, "Algorithms in Geometric Group Theory", PhD Thesis, University of California-Berkeley (1999).

[52] C. Gutiérrez, Satisfiability of Equations in Free Groups is in PSPACE, *32nd Ann. ACM Symp. Theory Comput. (STOC'2000)*, ACM Press 2000.

[53] V. Guba, On the embeddability of semigroups in groups, *Mat. Zametki* **54** (1994) 3-14 (in Russian).

[54] V. Guba, On the relationship between the word problem and the divisibility problems in one-relator semigroups, *Russian Math. Izv.* **61**, 3 (1997).

[55] V. Guba, V.S. Guba, The word problem for the relatively free semigroup satisfying $t^m = t^{m+n}$ with $m \geq 3$, *Inter. J. Algebra and Computation* **2** (3) (1993) 335-348.

[56] S. Haataja, S. Margolis and J. Meakin, Bass-Serre theory for groupoids and the structure of full regular semigroup amalgams, *J. Algebra* **183** (1996) 38-54.

[57] M. Hall Jr., A toplogy for free groups and related groups, *Ann. of Math.* (2) **52** (1950) 127-139.

[58] R. Hancock and I. Raeburn, The C^*-algebras of some inverse semigroups, *Bull. Australian Math. Soc.* **42** (1990) 335-348.

[59] K. Henckell, S. Margolis, J.-E. Pin and J. Rhodes, Ash's type-II theorem, profinite topology and Malcev products, *Inter. J. Algebra and Computation.* **1** (1991) 411-436.

[60] P. Higgins, "Notes on Categories and Groupoids", Vol. 32, Mathematical Studies, Van Nostrand Reinhold Company, London (1971).

[61] J.E. Hopcroft and J.D. Ullman, "Introduction to Automata Theory, Languages and Computation", Addison-Wesley, New York (1979).

[62] A.G. Howson, On the intersection of finitely generated free groups, *J. London Math. Soc.* **29** (1954) 428-434.

[63] S.V. Ivanov and J.C. Meakin, On asphericity and the Freiheitssatz for certain finitely presented groups, *J. Pure Appl. Algebra* 159 (2001) 113-121.

[64] S.V. Ivanov, S.W. Margolis and J.C. Meakin, On one-relator inverse monoids and one-relator groups, *J. Pure Appl. Algebra* 159 (2001), 83-111.

[65] T. Jajcayova, "*HNN* extensions of inverse semigroups", PhD Thesis, University of Nebraska-Lincoln (1997).

[66] J. James, (private communication, paper in preparation).

[67] P.R. Jones, S.W. Margolis, J. Meakin and J.B. Stephen, Free products of inverse semigroups II, *Glasgow Math. J.* **33** (1991) 373-387.

[68] I. Kapovich and A. Myasnikov, Stallings foldings and subgroups of free groups, *J. Algebra*, **248** (2002) 608-668.

[69] I. Kapovich, R. Weidman and A. Myasnikov, Foldings, graphs of groups and the membership problem, *Inter. J. Algebra and Computation*, Vol. 15, No. 1 (2005) 95-128.

[70] E.V. Kashintsev, Small cancellation conditions and embeddability of semigroups in groups, *Inter. J. Algebra and Computation* **2** (1992) 433-441.

[71] J. Kellendonk and M.V. Lawson, Tiling semigroups, *J. Algebra* **224** (2000) 140-150.

[72] C. Kilgour, Embedding monoids in groups, Preprint.

[73] K. Krohn and J. Rhodes, Algebraic theory of machines I: Prime decomposition theorem for finite semigroups and machines, *Trans. Amer. Math. Soc.* **116** (1965) 450-464.

[74] S. Krstic, Embedding semigroups in groups - a geometric approach, *Publ. Inst. Math. (Beograd) (N.S.)* **38** (52) (1995) 69-82.

[75] G. Lallement, On monoids presented by a single relation, *J. Algebra* 32 (1977) 370-388.

[76] M.V. Lawson, "Inverse Semigroups: the Theory of Partial Symmetries", World Scientific (1998).

[77] M.V. Lawson, The Möbius inverse monoid, *J. Algebra* **200** (1998) 428-438.

[78] S. Lindblad, "Inverse monoids presented by a single sparse relator", PhD Thesis, University of Nebraska (2003).

[79] A. A. Lorents, Representations of sets of solutions of systems of equations with one unknown in a free group, *Dokl. Akad. Nauk. SSSR* 178 (1968), 290–292 (Russian).

[80] M. Lothaire, *Algebraic Combinatorics on Words*, Cambridge University Press (2001).

[81] R. C. Lyndon, Equations in free groups, *Trans. Amer. Math. Soc.* 96 (1960), 445–457.

[82] R.C. Lyndon and P.E. Schupp, "Combinatorial Group Theory", Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 89, Springer-Verlag, Berlin (1977).

[83] G. S. Makanin, Equations in a Free Group, *Izv. Akad. Nauk. SSR, Ser. Math* 46 (1983) 1199-1273. English transl. in *Math. USSR Izv.* 21 (1983).

[84] G. S. Makanin, Problem of Solvability of Equations in Free Semigroup, *Math. Sbornik* 103 (1977) 147-236. English transl. in *Math. USSR Sbornik* 32 (1977).

[85] S. Margolis and J. Meakin, E-unitary inverse monoids and the Cayley graph of a group presentation, *J. Pure and Appl. Algebra* **58** (1989) 45-76.

[86] S. Margolis and J. Meakin, Free inverse monoids and graph immersions, *Int. J. Algebra and Computation*, Vol. 3, No. 1 (1993) 79-99.

[87] S. Margolis and J. Meakin, Inverse monoids, trees and context-free languages, *Trans Amer. Math. Soc.* **335** (1) (1993) 259-276.

[88] S. Margolis, J. Meakin and J. Stephen, Some decision problems for inverse monoid presentations, in "Semigroups and their applications" (eds. S.M. Goberstein, P.M. Higgins), Reidel, Dordrecht (1987) 99-110.

[89] S. Margolis, J. Meakin and J. Stephen, Free objects in certain varieties of inverse semigroups, *Canadian J. Math.* **42** (1990) 1084-1097.

[90] S. Margolis, J. Meakin and Z. Šuniḱ, Distortion functions and the membership problem for submonoids of groups and monoids, *Contemporary Mathematics* Vol. **372** (2005) 109-129.

[91] S. Margolis, M. Sapir and P. Weil, Closed subgroups in pro-**V** topologies and the extension problem for inverse automata, *Inter. J. Algebra and Computation* Vol 11, No. 4 (2001), 405-445.

[92] L. Markus-Epstein, "Algorithmic problems in subgroups of some finitely presented groups", PhD Thesis, Bar-Ilan University (2005).

[93] D.B. McAlister, Groups, semilattices and inverse semigroups, *Trans. Amer. Math. Soc.* **192** (1974) 227-244.

[94] D.B. McAlister, Groups, semilattices and inverse semigroups II, *Trans. Amer. Math. Soc.* **196** (1974) 251-270.

[95] D.B. McAlister and R. McFadden, The free inverse monoid on two commuting generators, *J. Algebra* 32 (1974) 215-233.

[96] J. McCammond, The solution to the word problem for the relatively free semigroups satisfying $t^a = t^{a+b}$ with $a \geq 6$, *Inter. J. Algebra and Computation* **1** (1991) 1-32.

[97] J. McCammond and D. Wise, Coherence, local quasiconvexity, and the perimeter of 2-complexes, preprint.

[98] R. McKenzie and R.J. Thompson, An elementary construction of unsolvable word problems in group theory, in Z. Boone et al. (Eds.), Word Problems, Studies in Logic, Vol. 71, North Holland, Amsterdam (1973).

[99] K.A. Mihailova, The occurrence problem for free products of groups, Mat. Sb. **75**(117) (1968) 199-210.

[100] W. D. Munn, Free inverse semigroups, *Proc. London Math. Soc.* (3) 29 (1974), 385-404.

[101] A.I. Mal'cev, On the immersion of an algebraic ring into a field, *Math. Ann.* **133** (1937) 686-691.

[102] A.I. Mal'cev, On the immersion of associative systems in groups, *Mat. Sbornik (N.S.)* 6 (1939), 331-336.

[103] A Markov, Impossibility of algorithms for recognizing some properties of associative systems, *Dokl. Acad. Nauk SSSR* 77 (1951) 953-956.

[104] A. Nica, On a groupoid construction for actions of certain inverse semigroups, *International J. Math.* Vol.5, No 3 (1994) 349-372.

[105] M. Nivat and J-F. Perrot, Une généralisation du monoïde bicyclique, *Comptes Rendus de l'Academie des Sciences de Paris* **271** (1970), 824-827.

[106] J. Okninski, "Semigroups of matrices", *World Scientific* (1998).

[107] A.Yu. Ol'shanskii, "Geometry of Defining Relations in Groups", Nauka, Moscow (1989), English translation in: *Math. and its Applications (Soviet series)*, Vol. 70, Kluwer Dordrecht (1991).

[108] L. Paris, Artin monoids inject in their groups, *Comment. Math. Helv.* **77** (2002) no. 3, 609-637.

[109] L. Paris, The proof of Birman's conjecture on singular braid monoids, *Geom. Topol* **8** (2004) 1281-1300 (electronic).

[110] Alan L.T. Paterson, Groupoids, Inverse Semigroups, and their Operator Algebras, *Birkhäuser* (1998).

[111] M. Petrich, "Inverse semigroups", Wiley (1984).

[112] J.-E. Pin, "Varieties of Formal Languages", Plenum, London (1986).

[113] W. Plandowski, Satisfiability of Word Equations with Constants is in PSPACE, *Proc. 40th Ann. Symp. Found. Comput. Sci. (FOCS'99), IEEE Computer Society Press*, (1999) 495-500.

[114] M. Putcha, Linear Algebraic Monoids, *London Math Soc Lecture Note Series*, **133**, Cambridge Univ Press, Cambridge, UK (1988).

[115] M. O. Rabin, Decidability of Second Order Theories and Automata on Infinite Trees, *Trans. Amer. Math. Soc.* 141 (1969) 1-35.

[116] A. A. Razborov, On Systems of Equations in a Free Group, *Math. USSR-Izv.* 25 (1985) 115-162.

[117] D. Rees, On semigroups, *Proc. Cambridge Phil. Soc.* 36 (1940) 387-400.

[118] J.H. Remmers, On the geometry of semigroup presentations, *Adv. in Math.* **36** (3) (1980) 283-296.

[119] L. Ribes and P.A. Zalesskiǐ, On the profinite topology on a free group, *Bull. London Math. Soc.* **25** (1993) 37-43.

[120] L. Ribes and P.A. Zalesskiǐ, The pro-p topology of a free group and algorithmic problems in semigroups, *Inter. J. Algebra and Computation* **4** (1994) 359-374.

[121] B. V. Rozenblat, Diophantine theories of free inverse semigroups, *Sibirskii Mat. Zhurnal* (6) 26 (1986), 101–107 (Russian); english transl. in pp. 860–865.

[122] R.L. Ruyle, "Pseudovarieties of Inverse Monoids", PhD Thesis, University of Nebraska-Lincoln (1997).

[123] M.V. Sapir, Algorithmic problems for amalgams of finite semigroups, *J. Algebra* **229** (2000), No.2, 514-531.

[124] O. Sarkesian, On identity and divisibility problems in semigroups and groups without cycles, *Izv. Akad. Nauk SSSR, Ser. Mat.* **45**, 6 (1981) 1424-1441.

[125] H.E. Scheiblich, Free inverse semigroups, *Proc. Amer. Math. Soc.* **38** (1973), 1-7.

[126] B.M. Schein, Representations of generalized groups, *Izv. Vyss. Ucebn. Zav. Matem.* issue **3** (1962) 164-176 (Russian).

[127] P. Schupp, Coxeter groups, 2-completion, perimeter reduction and subgroup separability, *Geom. Dedicata* **96** (2003) 179-198.

[128] J.P. Serre, "Trees", Springer-Verlag, New York (1980).

[129] P. V. Silva, Word Equations and Inverse Monoid Presentations, in *Semigroups and Applications, Including Semigroup Rings*, ed. S. Kublanovsky, A. Mikhalev, P. Higgins, J. Ponizovskii, "Severny Ochag", St. Petersburg (1999).

[130] J.R. Stallings, A graph-theoretic lemma and group embeddings, in "Combinatorial Group Theory and Topology", ed. S. Gersten and J. Stallings, Princeton University Press (1987) pp 145-155.

[131] J. Stallings, Adian groups and pregroups, in "Essays in group theory", vol 8 of *Math. Sci. Res. Inst. Publ.*, pp. 321-342, Springer, New York (1987).

[132] J.R. Stallings, "Group Theory and Three-Dimensional Manifolds", Yale Math. Monograph **4**, Yale Univ. Press (1971).

[133] J.R. Stallings, Topology of finite graphs, *Invent. Math.* **71**, No. 3 (1983) 551-565.

[134] J. Stillwell, "Classical Topology and Combinatorial Group Theory", Springer-Verlag, New York (1980).

[135] B. Steinberg, Fundamental groups, inverse Schützenberger automata, and monoid presentations, *Comm. Algebra* **28** (2000) 5235-5253.

[136] B. Steinberg, Inverse automata and profinite topologies on a free group, *J. Pure Appl. Algebra*, **167** (2002), No. 2-3, 341-359.

[137] B. Steinberg, Inevitable graphs and profinite topologies: some solutions to algorithmic problems in monoid and automata theory stemming from group theory, *Inter. J. Algebra and Computation* Vol. 11, No. 1 (2001) 25-71.

[138] B. Steinberg, A topological approach to inverse and regular semigroups, *Pacific J. Math* **208** (2) (2003) 367-396.

[139] J.B. Stephen, Presentations of inverse monoids, *J. Pure and Appl. Algebra* **63** (1990) 81-112.

[140] J.B. Stephen, Inverse monoids and rational subsets of related groups, *Semigroup Forum* 46 (1) (1993) 98-108.

[141] H. Straubing, "Finite automata, formal logic, and circuit complexity", Birkhäuser Boston Inc., Boston, MA. (1994).

[142] A. Suschkewitsch, Über die endlichen Gruppen ohne das Gesetz der eindeutigen Umkehrbarkeit, *Math. Ann.* 99 (1928) 30-50.

[143] M. Takahasi, Note on chain conditions in free groups, *Osaka Math. J* **3** (1951) 221-225.

[144] G. Watier, On the word problem for single relation monoids with an unbordered relator, *Int. J. Algebra and Computation*, Vol 7, No. 6 (1997) 749-770.

[145] P. Weil, Profinite methods in semigroup theory, *Inter. J. Algebra and Computation* Vol. 12, Nos. 1 & 2 (2002), 137-178.

[146] A. Yamamura, HNN extensions of inverse semigroups and applications, *Inter. J. Algebra and Computation* Vol. 7, No 5 (1997) 605-624.

[147] A. Yamamura, HNN-extensions of semilattices, *Inter. J. Algebra and Computation* **9** (1999) 555-596.

[148] L. Zhang, A short proof of a theorem of Adian, *Proc. Amer. Math. Soc.* Vol 116, No. 1 (1992) 1-3.

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